Number 1 - September 2020 Problems

October 17, 2020

Problems

Problem 1A. Proposed by Vedaant Srivastava

Let x and y be positive reals such that $x+y=6$. Find the minimum value of

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)$$

and determine the values of x, y which yield such a minimum.

Solution:

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)=xy+\frac{1}{xy}+\frac{(x+y)^2-2xy}{xy}=xy+\frac{37}{xy}-2\geq 2(\sqrt{37}-1)$$

where the last inequality follows from AM-GM, satisfying equality when $xy=\sqrt{37}$.

Thus we have that $xy=\sqrt{37}$ and $x+y=6$.

Solving the system of equations yields that

$$\boxed{(x,y)=\left(3+\sqrt{9-\sqrt{37}},3-\sqrt{9-\sqrt{37}}\right)}$$

in some order.

Problem 2A. Proposed by Max Jiang

Find the largest $x\in\mathbb{N}$ such that 2020^x divides 202020!.

Solution:
It is known that the number of factors of a prime p that divides $n!$ is $\sum_{i=1}^{\infty}\left\lfloor\frac{n}{p^i}\right\rfloor$.

The prime factorization of 2020 is $2^2 \cdot 5 \cdot 101$. We have

$$\begin{aligned}\sum_{i=1}^{\infty}\left\lfloor\frac{202020}{101^i}\right\rfloor &= 2000+19\\ &= 2019,\\ \sum_{i=1}^{\infty}\left\lfloor\frac{202020}{5^i}\right\rfloor &> 40404\\ &> 2019, \text{ and}\\ \sum_{i=1}^{\infty}\left\lfloor\frac{202020}{2^i}\right\rfloor &> 101010\\ &> 2\cdot 2019.\end{aligned}$$

Thus, we have $(2^2)^{2019}|202020!$, $5^{2019}|202020!$, and $101^{2019}|202020!$ but $101^{2020} \nmid 202020!$, so $\boxed{x = 2019}$.

Problem 3A. Proposed by Nicolas Sullivan

In December 2020, an alien civilization is discovered. In trying to communicate, we discover that they use a mathematical operation $\star$, and they only tell us that $a \star a = 1$ and $a \star (bc) = a \star b + a \star c$, for any positive real a, b, c, $a \neq 1$. Find

$$\sum_{n=1}^{2019}\left[2020 \star \left(1+\frac{1}{n}\right)\right].$$

Solution:

First, we need to establish some properties of the operation $\star$. By the second property, setting $c = 1$, we know that $a \star (b(1)) = a \star b + a \star 1$, so $a \star 1 = a \star b - a \star b = 0$ for any valid a, b.

Next, using this result and the second property again, we can set $c = \frac{1}{b}$, since $b \neq 0$. As a result, $a \star (b(\frac{1}{b})) = a \star b + a \star \frac{1}{b}$, so $a \star \frac{1}{b} = a \star 1 - a \star b = -a \star b$ for any valid a, b.

Now we can examine the sum, which we shall denote S. The sum can be expressed as

$$S = \sum_{n=1}^{2019}\left[2020 \star \left(\frac{n+1}{n}\right)\right],$$

and by the second property of the $\star$ operation, this can be decomposed to

$$S = \sum_{n=1}^{2019}\left[2020 \star (n+1) + 2020 \star \left(\frac{1}{n}\right)\right].$$

This can be then written as

$$S = \sum_{n=1}^{2019}\left[2020 \star (n+1) - 2020 \star (n)\right],$$

which telescopes to

$$S = 2020 \star (2019 + 1) - 2020 \star 1.$$

Since $a \star 1 = 0$ and $a \star a = 1$ for any valid a, then this sum can be simplified to $\boxed{S = 2020 \star 2020 - 2020 \star 1 = 1.}$

Problem 4A. Proposed by Ken Jiang

Alice and Bob are playing a game. They take turns drawing cards from separate 52-card decks. After her turn, Alice places her card back into her deck and reshuffles it, while Bob does not. The player who draws the ace of spades first is the winner. If Alice goes first, what is the probability that she will win?

Solution:

Since the ace of spades is equally likely to be in any position in the deck, the probability that Bob draws it on his nth turn is always $\frac{1}{52}$. The probability that Alice has not drawn the ace of spades in n turns is $\frac{51^n}{52^n}$. Thus, the probability that Bob will win is $\frac{1}{52}\sum_{i=1}^{52}\frac{51^i}{52^i}$, which evaluates to $\frac{51\left(52^{52}-51^{52}\right)}{52^{53}}$. The chance that Alice wins is the complement of this, which is $1 - \frac{51\left(52^{52}-51^{52}\right)}{52^{53}} = \frac{52^{52}+51^{53}}{52^{53}} = \frac{1}{52} + \left(\frac{51}{52}\right)^{53}$

Problem 5A. Proposed by Alexander Monteith-Pistor

Let $\triangle ABC$ be an isosceles right triangle with right angle at A and incenter I. The point P is randomly chosen inside $\triangle ABC$. Find the probability that $\triangle PAI$ is an acute triangle (all of its angles are acute).

Problem 6A. Proposed by Andy Kim

Prove that all six-digit palindromes are divisible by 11.

Solution:

We represent the six-digit palindrome as $\overline{abccba}$.

Then,

$$\overline{abccba} = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + c \cdot 10^2 + b \cdot 10^1 + a$$

Noting that $10 \equiv -1 \pmod{11}$, we have

$$\begin{aligned}
& a\cdot 10^5 + b\cdot 10^4 + c\cdot 10^3 + c\cdot 10^2 + b\cdot 10^1 + a && \pmod{11}\\
\equiv\ & a\cdot(-1)^5 + b\cdot(-1)^4 + c\cdot(-1)^3 + c\cdot(-1)^2 + b\cdot(-1)^1 + a && \pmod{11}\\
\equiv\ & a\cdot(-1) + b\cdot 1 + c\cdot(-1) + c\cdot 1 + b\cdot(-1) + a && \pmod{11}\\
\equiv\ & -a + b - c + c - b + a && \pmod{11}\\
\equiv\ & 0 && \pmod{11}
\end{aligned}$$

Therefore, $\overline{abccba}$ is divisible by 11.

Problem 7A. Proposed by Kevin Li

If $x^2 + \frac{y^2}{4} = 8x + y + 8$, find the largest value of $5x + 6y$.

Solution:

$$\begin{gathered}
x^2 + \frac{1}{4}y^2 = 8x + y + 8\\
x^2 - 8x + 16 + \frac{1}{4}y^2 - y + 1 = 25\\
\frac{5^2(x-4)^2}{5^2} + \frac{12^2(\frac{1}{2}y-1)^2}{12^2} = 5^2\\
\frac{(5x-20)^2}{5^2} + \frac{(6y-12)^2}{12^2} = 5^2\\
5^2 \geq \frac{(5x+6y-32)^2}{5^2+12^2}\\
5^2(13)^2 \geq (5x+6y-32)^2\\
65 \geq 5x + 6y - 32\\
97 \geq 5x + 6y
\end{gathered}$$

Problem 8A. Proposed by DC

The area of the triangle ABC is S and the altitude to a is h_a. Prove that the following relationship is true:

$$\frac{a}{h_a^2bc} + \frac{b}{h_a^2ac} + \frac{c}{h_a^2ab} = \frac{a(a^2+b^2+c^2)}{4S^2bc}.$$

Solution:

In any triangle $ah_a = 2S$. Consequently:

$$\frac{a}{h_a^2bc} + \frac{b}{h_a^2ac} + \frac{c}{h_a^2ab} = \frac{a^3}{4S^2bc} + \frac{ba^2}{4S^2ac} + \frac{ca^2}{4S^2ab} = \frac{a^3}{4S^2bc} + \frac{b^2a^2}{4S^2abc} + \frac{c^2a^2}{4S^2abc} = \frac{a(a^2+b^2+c^2)}{4S^2bc}.$$

Problem 9A. Proposed by DC

In any triangle ABC, prove that

$$b^2 sin^2 C + c^2 sin^2 B = 2bc(cosA + cosB \cdot cosC).$$

Solution:
$cosA + cosB \cdot cosC = -cos(B - c) + cosB \cdot cosC = sinB \cdot sinC.$
The relationship to be proved becomes:
$b^2 \cdot sin^2 C + c^2 \cdot sin^2 B = 2bc \cdot sinB \cdot sinC$
or $b^2 \cdot sin^2 C - 2bc \cdot sinB \cdot sinC + c^2 \cdot sin^2 B = 0$
or $(b \cdot sinC - c \cdot sinB)^2 = 0.$

Problem 10A. Proposed by DC

Find all three digit numbers $\overline{abc}$ such that the last three digits of $(\overline{abc})^3$ are $\overline{abc}$.

Solution:
If $(\overline{abc})^3$ is ending in $\overline{abc}$ then $(\overline{abc})^3 - \overline{abc}$ has 1000 as divisor.
In addition, $(\overline{abc})^3 - \overline{abc}$=$(\overline{abc} - 1)\overline{abc}(\overline{abc} + 1) = 1000k = 125 \cdot 8 \cdot k.$
These are three consecutive numbers. Only one of them must have 5 as divisor. Consequently, only one of them can have 125 as divisor.
We have now three cases:
Case 1. $(\overline{abc} - 1)$ has 125 as divisor.
Case 2. $\overline{abc}$ has 125 as divisor.
Case 3. $(\overline{abc} + 1)$ has 125 as divisor.

Let us solve each of them:
Case 1. $(\overline{abc} - 1)$ has 125 as divisor.
The three digit multiple by 125 are: 125, 250, 375, 500, 625, 750 and 875. Thus, $\overline{abc}$ can be: 126, 251, 376, 501, 626, 751 and 876.
In addition to this condition, $(\overline{abc})^3 - \overline{abc}$ has 8 as divisor.
If $(\overline{abc} - 1)$ is odd, then $(\overline{abc} + 1)$ is odd, and consequently $\overline{abc}$ must be divisible to 8. Only 376 satisfies this condition.
If $(\overline{abc} - 1)$ is even, then $(\overline{abc} + 1)$ is even, and their product is divisible by 8. Consequently all odd values for $\overline{abc}$ are solutions, namely 251, 501, 751.

Case 2. $\overline{abc}$ has 125 as divisor.
The even multiples of 125 are not suitable because $(\overline{abc} - 1)$ and $(\overline{abc} + 1)$ will be even and 250, 500, or 750 don't have 8 as divisor. The odd multiples are good solutions, providing $(\overline{abc} - 1)$ and $(\overline{abc} + 1)$ as even. Consequently, 125, 375, 625 and 875 are solutions.

Case 3. $(\overline{abc} + 1)$ has 125 as divisor.
As in Case 1, the odd values of $\overline{abc}$ are suitable. We will have 249, 499, 749 as solutions. In addition, 1000 is multiple by 125 and 999 is also a solution. If $\overline{abc}$ is even, must be multiple by 8. From the set of 124, 374, 624 and 874, only 624

is multiple by 8. This is also a solution.

To conclude, there are 13 solutions: 125, 249, 251, 375, 376, 499, 501, 624, 625, 749, 751, 875 and 999.

Problem 1B. Proposed by Vedaant Srivastava

Johnny the sketchy vendor has infinitely many stuffed toys lined up in a row. From left to right, he has 1 teddy bear costing \$1, 2 teddy bears costing \$2, 3 teddy bears costing \$3, and so on.

Show that the price of the n-th teddy bear in the row from the left is

$$\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$$

dollars.

Solution:
Let k denote the price of the n-th teddy bear. Thus we have that

$$\begin{gathered} k(k-1)/2 < n \leq k(k+1)/2 \\ k^2 - k < 2n \leq k^2 + k \\ k^2 - k + \tfrac{1}{4} < 2n < k^2 + k + \tfrac{1}{4} \\ k - \tfrac{1}{2} < \sqrt{2n} < k + \tfrac{1}{2} \\ k < \sqrt{2n} + \tfrac{1}{2} < k + 1 \\ \boxed{k = \left\lfloor \sqrt{2n} + \tfrac{1}{2} \right\rfloor} \end{gathered}$$

Problem 2B. Proposed by Max Jiang

Compute

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}.$$

Solution:
Let

$$S = \sum_{k=1}^{\infty} \frac{k^2}{2^k}.$$

Then,

$$2S = \sum_{k=1}^{\infty} \frac{k^2}{2^{k-1}}$$

so

$$\begin{aligned} 2S - S &= \sum_{k=1}^{\infty} \frac{k^2}{2^{k-1}} - \sum_{k=1}^{\infty} \frac{k^2}{2^k} \\ S &= 1 + \sum_{k=1}^{\infty} \left(\frac{(k+1)^2}{2^k} - \frac{k^2}{2^k} \right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{2k+1}{2^k}. \end{aligned}$$

Now,

$$2S = 2 + \sum_{k=1}^{\infty} \frac{2k+1}{2^{k-1}}$$

so

$$\begin{aligned} 2S - S &= 2 + \sum_{k=1}^{\infty} \frac{2k+1}{2^{k-1}} - \left(1 + \sum_{k=1}^{\infty} \frac{2k+1}{2^k} \right) \\ S &= 1 + 3 + \sum_{k=1}^{\infty} \left(\frac{2(k+1)+1}{2^k} - \frac{2k+1}{2^k} \right) \\ &= 4 + \sum_{k=1}^{\infty} \frac{2}{2^k} \\ &= 4 + 2 \\ &= \boxed{6}. \end{aligned}$$

Problem 3B. Proposed by Nicolas Sullivan

Let F_n be the nth number in the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for any natural n. If $\varphi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$, show that for any $n \in \mathbb{Z}^+$,

$$27F_{n-1}F_nF_{n+1} < (\varphi^{-1}F_{n-1} + \varphi F_{n+1})^3.$$

Solution:

First, we recognize the usual relation $\varphi^{-1} = \varphi - 1$ because

$$\begin{aligned} \varphi^{-1} &= \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \\ &= \lim_{n\to\infty} \frac{F_{n+1} - F_n}{F_n} \\ &= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} - 1 \\ &= \varphi - 1. \end{aligned}$$

By the AM-GM inequality, we know that

$$(F_{n-1}F_nF_{n+1})^{1/3} \leq \frac{\varphi F_{n-1} + F_n + \varphi^{-1}F_{n+1}}{3}.$$

Equality only holds if $\varphi F_{n-1} = F_n = \varphi^{-1}F_{n+1}$. Since φ is irrational, $\varphi F_{n-1} \neq F_n$, and thus equality cannot hold,

$$(F_{n-1}F_nF_{n+1})^{1/3} < \frac{\varphi F_{n-1} + F_n + \varphi^{-1}F_{n+1}}{3}.$$

Thus, we can rearrange the above inequality to yield

$$27F_{n-1}F_nF_{n+1} < \left(\varphi F_{n-1} + F_n + \varphi^{-1}F_{n+1}\right)^3.$$

Using the definition of the Fibonacci sequence, we have that $F_n = F_{n+1} - F_{n-1}$, so

$$27F_{n-1}F_nF_{n+1} < \left((\varphi - 1)F_{n-1} + (\varphi^{-1} + 1)F_{n+1}\right)^3.$$

Finally, since $\varphi^{-1} = \varphi - 1$,

$$27F_{n-1}F_nF_{n+1} < \left(\varphi^{-1}F_{n-1} + \varphi F_{n+1}\right)^3.$$

Problem 4B. Proposed by Ken Jiang

Alice and Bob are playing a new game. Starting from N, they take turns counting down F_i numbers, where F_i must be a member of the Fibonacci sequence. Alice goes first, and the player who counts to 1 is the winner. Show that there are infinite values of N such that, no matter how Alice plays, Bob can win.

Problem 5B. Proposed by Proposed by Alexander Monteith-Pistor

Let $S = \{0, 1, 2, ..., 2020\}$ and $f : S \to S$ satisfy

$$f(x)f(y)f(xy) = f(f(x + y))$$

for all $x, y \in S$ with $xy \leq 2020$ and $x + y \leq 2020$. Find the maximum possible value of

$$\sum_{i=0}^{2020} f(i)$$

Problem 6B. Proposed by Nikola Milijevic

Prove that there exist infinitely many positive integers n such that $3^n + 2$ and $5^n + 2$ are both composite.

Solution:
Let i be any non-negative integer. Using Fermat's Little Theorem, we can get that

$$3^{10} \equiv 3^{10i} \equiv 1 \pmod{11}$$
$$3^{10i+2} \equiv 9 \pmod{11}$$
$$3^{10i+2} + 2 \equiv 0 \pmod{11}$$

Similarily, let j be any non-negative integer. Using Fermat's Little Theorem,

$$5^2 \equiv 5^{2j} \equiv 1 \pmod{3}$$
$$5^{2j} + 2 \equiv 0 \pmod{3}$$

As $10i + 2$ will always be an even number, we can conclude that when n is of form $10k + 2$, where k is any non-negative integer, both $5^n + 2$ and $3^n + 2$ will be composite.

Problem 7B. Proposed by Nikola Milijevic

Prove that equation $x^5 - x = 3 - y^4$ has no solutions in integers x and y.

Solution:
We show this is impossible by comparing congruence modulo 5 of both the left-hand and right-hand sides.

Using the Corollary to Fermat's Little Theorem, we can see that

$$x^5 - x \equiv x - x \equiv 0 \pmod{5}$$

Hence, the left hand side is always congruent to 0 modulo 5. We now look at two cases, when $5 \mid y$ and when $5 \nmid y$.

<u>Case 1:</u> If $5 \mid y$, it follows that

$$3 - y^4 \equiv 3 \pmod{5}$$

<u>Case 2:</u> If $5 \nmid y$, then we are able to use Fermat's Little Theorem

$$3 - y^4 \equiv 3 - (1)^4 \equiv 2 \pmod{5}$$

Therefore, as the left-hand and right-hand sides never have the same congruence modulo 5, x and y cannot have integer solutions.

Problem 8B. Proposed by Andy Kim

Let a, b, and c be positive real numbers. Given that $abc = 1$, prove that

$$\frac{a^3 - 1}{b^2c^2} + \frac{b^3 - 1}{c^2a^2} + \frac{c^3 - 1}{a^2b^2} \geq 0$$

Problem 9B. Proposed by Kevin Li

There are n students at Main High School. Each student is friends with at least $\lceil \frac{n}{k} \rceil$ other students (friendships are mutual),for some positive integral k. Define a *friend chain* to be a series of students a_1,a_2,a_3,...,a_{i-1},a_i where i is a positive integer, such that a_1 is friends with a_2, a_2 is friends with a_3, and so on until a_{i-1} is friends with a_i. We call a *friend group* a set of students such that there exists a friend chain between every two students in the set. No student outside of the friend group is friends with any members of the friend group, and any student who has a friend chain to a member of a friend group is also part of that friend group. Show that there cannot exist k or more friend groups.

Solution:

We first show that, for two friend groups A and B, where $A \neq B$, $A \cap B = \varnothing$. Assume for contradiction that $A \cap B \neq \varnothing$, that is, there exists some student a such that $a \in A$ and $a \in B$. If a is the only member of both sets, $A = B$. Without loss of generality, let A have more than one element. Let another element of A be c. Then, there exists a friend chain from a to c. Then, there is a friend chain from b to c, namely b, a,...,c. Since every member of B has a friend chain to b, there exists a friend chain from every element of B to c, meaning A and B are the same set.

Next, assume for contradiction there are k or more friend groups. Note that every friend group has at minimum $\lceil \frac{n}{k} \rceil + 1$ students, since a student has at minimum $\lceil \frac{n}{k} \rceil$ friends. Since every friend group is mutually exclusive, the minimum number of students is

$$(\lceil \frac{n}{k} \rceil + 1)k = (\lceil \frac{n}{k} \rceil)k + k \geq \frac{n}{k}k + k = n + k$$

which is a contradiction.

Problem 10B. Proposed by DC

Find the values for m such that the equation

$$x^3 - 2mx^2 + 3x + 9 = 0$$

has one double root.

Solution:

Let us denote the roots as a and b, the first being the double root.
The first part of the equation becomes:

$$x^3 - 2mx^2 + 3x + 9 = (x - a)^2(x - b)$$

or

$$x^3 - 2mx^2 + 3x + 9 = x^3 - (2a + b)x^2 + (a^2 + 2ab)x - a^2b$$

Three additional relationships will be derived:

R1: $2a + b = 2m$

R2: $a^2 + 2ab = 3$
R3: $a^2 b = -9$

From **R3**, substitute b in **R2**:
$a^2 - 2a\frac{9}{a^2} = 3$
$a^3 - 3a - 18 = 0$
$(a-3)(a^3 + 3a + 6) = 0$
with only one real root for $a = 3$. The equation $a^3 + 3a + 6 = 0$ has complex roots.
With $a = 3$, the value $b = -1$ can be derived from **R3**.
Finally, from **R1**: m=2.5
The equation becomes:

$$x^3 - 5x^2 + 3x + 9 = 0$$

with $a = 3$ as double root and $b = -1$ as simple root.