Year 2 - Number 1 - September 2021 Problems

November 11, 2021

Problems

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 53A. Proposed by Nikola Milijevic

The positive integers $a_1, a_2, \ldots, a_n$ are not greater than 2021, with the property that $\text{lcm}(a_i, a_j) > 2021$ for all $i, j, i \neq j$. Show that:

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2$$

Solution Problem 53A

We let k_1 be the number of a_i in the interval $(\frac{2021}{2}, 2021]$, k_2 the number of a_i in the interval $(\frac{2021}{3}, \frac{2021}{2}]$, and so on. Note if we have $\frac{2021}{m+1} < a_i \leq \frac{2021}{m}$ for some m and i, then $a_i, 2a_i, \ldots, ma_i$ are all no greater than 2021. Since $\text{lcm}(a_i, a_j) > 2021$, $k_1 + 2k_2 + 3k_3 + \ldots$ is the number of distinct integers no greater than 2021, that are multiples of one of the a_i. We have:

$$\begin{aligned} 2k_1 + 3k_2 + 4k_3 + \ldots &= (k_1 + k_2 + k_3 + \ldots) + (k_1 + 2k_2 + 3k_3 + \ldots) \\ &\leq n + 2021 \\ &\leq 4042 \end{aligned}$$

Finding an upper bound for the summation,

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{a_i} &< k_1\frac{2}{2021}+k_2\frac{3}{2021}+k_3\frac{4}{2021}+\dots\\ &= \frac{2k_1+3k_2+4k_3+\dots}{2021}\\ &\le \frac{4042}{2021}\\ &= 2\end{aligned}$$

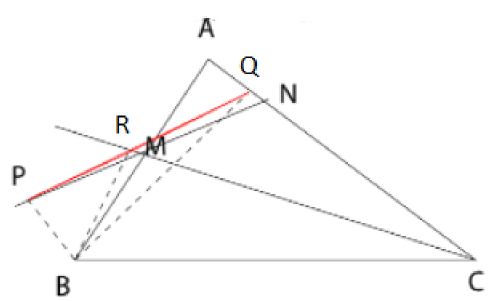
Therefore,

$$\sum_{i=1}^{n}\frac{1}{a_i} < 2$$

Problem 55A. Proposed by Eliza Andreea Radu

Consider the triangle ABC with $AB = 4$ cm, $BC = 6$ cm and $AC = 5$ cm. Take $M \in (AB)$ and $N \in (AC)$ such that $\cos(\measuredangle AMN) = \frac{3}{4}$. The feet of the perpendiculars drawn from B to MN, NC, and MC are P, Q, and R, respectively. What does P, Q, R form?

Solution Problem 55A



Applying Cosine Law in $\triangle ABC$, we obtain:

$$\cos(\measuredangle ACB) = \frac{BC^2 + AC^2 - AB^2}{2 \cdot BC \cdot AC} = \frac{6^2 + 5^2 - 4^2}{2 \cdot 6 \cdot 5} = \frac{3}{4}.$$

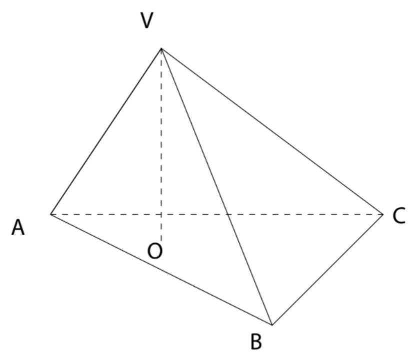
Since we are given that $\cos(\measuredangle AMN) = \frac{3}{4}$, we know that $\cos(\measuredangle ACB) = \cos(\measuredangle AMN)$, meaning that $\measuredangle ACB = \measuredangle AMN$. Therefore, $BMNC$ is cyclic.

We are given that $BP \perp MN$, $BR \perp CM$, and $BQ \perp CN$ (see above figure). Since B lies on the circumcircle of triangle MNC, we see that P, Q, R form Simson's Line, meaning that points P, R, and Q are collinear. Therefore, P, Q, and R form a line.

Problem 57A. Proposed by Eliza Andreea Radu

Consider the tetrahedron $VABC$ with a volume equal to 4 such that $m(\measuredangle ACB) = 45°$ and $\frac{AC+3\sqrt{2}(BC+VB)}{\sqrt{18}} = 6$. Find the distance from B to the plane (VAC).

Solution Problem 57A



First we build VO, the altitude from V.

We have that

$$\begin{aligned} V_{VABC} &= \frac{A_{\text{base}} \cdot h}{3} \\ &= \frac{A_{ABC} \cdot VO}{3} \\ &= \frac{\frac{AC \cdot BC \cdot \sin(45°)}{2} \cdot VO}{3} \\ &= \frac{AC \cdot BC \cdot \frac{\sqrt{2}}{2} \cdot VO}{6} \\ &= \frac{\frac{AC\sqrt{2}}{6} \cdot BC \cdot VO}{2}. \end{aligned}$$

Since $V_{VABC} = 4$, we have that

$$\frac{\frac{AC\sqrt{2}}{6} \cdot BC \cdot VO}{2} = 4 \Rightarrow \frac{AC\sqrt{2}}{6} \cdot BC \cdot VO = 8.$$

Because VO is the altitude, we know that $VO \perp (ABC)$. Since $OB \subset (ABC) \Rightarrow VO \perp OB$. If $O \neq B$, then $VO < VB$ via the relationship between the hypotenuse and legs in a right angled triangle. If $O = B$, then $VO = VB$. To conclude: $VO \leq VB$.

We are given that

$$\frac{AC + 3\sqrt{2}(BC + VB)}{\sqrt{18}} = \frac{AC}{3\sqrt{2}} + \frac{3\sqrt{2}(BC + VB)}{3\sqrt{2}} = \frac{AC\sqrt{2}}{6} + BC + VB = 6.$$

From $VB \geq VO$, we have

$$6 = \frac{AC\sqrt{2}}{6} + BC + VB \geq \frac{AC\sqrt{2}}{6} + BC + VO.$$

Through AM-GM, we get that

$$\frac{\frac{AC\sqrt{2}}{6} + BC + VO}{3} \geq \sqrt[3]{\frac{AC\sqrt{2}}{6} \cdot BC \cdot VO} = \sqrt[3]{8} = 2.$$

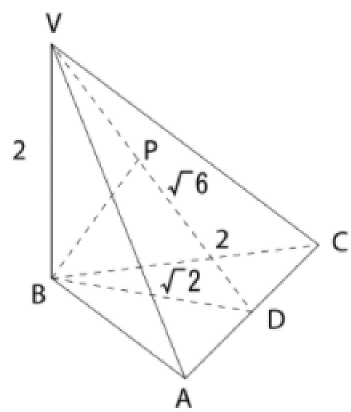
Thus,

$$\frac{AC\sqrt{2}}{6} + BC + VO \geq 6.$$

Consequently, $\frac{AC\sqrt{2}}{6} + BC + VO = 6$. This equality is satisfied only when $\frac{AC\sqrt{2}}{6} = BC = VO = \frac{6}{3} = 2$ and $VO = VB$.

Therefore, $AC = 6\sqrt{2}$, $BC = VO = VB = 2$, and $VB \perp (ABC)$.

Take $D \in AC$ such that $BD \perp AC$ and take $P \in VD$ such that $BP \perp VD$ (see figure below).



Because $VB \perp (ABC)$ and $AC \subset (ABC)$, therefore $VB \perp AC$.

Since $AC \perp VB, AC \perp BD, BD \cap VB = \{B\}$, and $BD, VB \subset (VBD)$, we have that $AC \perp (VBD)$.

Because $AC \perp (VDB)$ and $BP \subset (VBD)$, we get that $AC \perp BP$.

From the fact that $BP \perp AC, BP \perp VD, VD \cap AC = \{D\}$, and $VD, AC \subset (VAC)$, we know that $BP \perp (VAC)$. Thus,

$$d(B, (VAC)) = BP.$$

Since $BD \perp AC$ and $\measuredangle ACB = 45°$, we see that $\triangle BDC$ is a right-angled isosceles triangle, meaning that

$$BD = \frac{BC}{\sqrt{2}} = \sqrt{2}.$$

Because $VB \perp (ABC)$ and $BD \subset (ABC)$, we get that $VB \perp BD$, meaning that VDB is a right-angled triangle.

Applying Pythagorean Theorem to $\triangle VDB$ gives us

$$VD^2 = VB^2 + BD^2 = 2^2 + (\sqrt{2})^2 = 6 \Rightarrow VD = \sqrt{6}.$$

Since $2A_{VBD} = VB \cdot BD = VD \cdot BP$, we have that

$$BP = \frac{VB \cdot BD}{VD} = \frac{2 \cdot \sqrt{2}}{\sqrt{6}} = \frac{2\sqrt{3}}{3}.$$

Therefore, $d(B,(VAC)) = \frac{2\sqrt{3}}{3}$.

Problem 66A. Proposed by Adelina Sofian

Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 12$. Find the range of values for x, y, z such that

$$\frac{8 + 6x^2}{x^2 + 12} + \frac{8 + 6y^2}{y^2 + 12} + \frac{8 + 6z^2}{z^2 + 12} \leq 6.$$

Problem 67A. Proposed by Eliza Andreea Radu

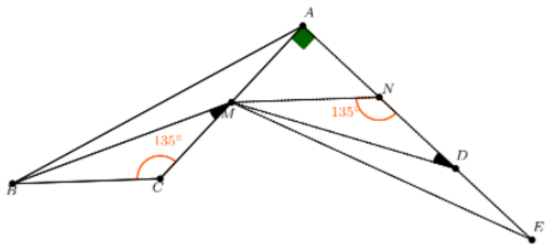
Consider the convex quadrilateral $ABCD$ and the parallelograms $ACPD$ and $ABDQ$. Find $m(\measuredangle(AC, BD))$ knowing that $BP = 16$, $CQ = 12$, $AC = 4$, and $BD = 7\sqrt{2}$.

Problem 68A. Proposed by Mihai Teodor Stupariu

In a given triangle AMB whose side lengths AM, MB, and AB are in the ratio of $1 : \sqrt{5} : \sqrt{10}$, take C on ray $(AM$ such that $m(\measuredangle(ACB)) = 135°$. Find $\frac{BC}{AM}$.

Solution Problem 68A

Take D such that $AD \perp AM$ and $2AM = AD$ and N middle of AD. Take E the symmetric point of N over D.



Knowing that $AM = a$, then $BM = \sqrt{5}a$ and $AB = \sqrt{10}a$.
Also, $AD = 2a$ and $AE = 3a$. From Pythagorean Theorem, $MD = \sqrt{a^2 + 4a^2} = \sqrt{5}a$ and $ME = \sqrt{10}a$.
We also have $\triangle EDM \equiv \triangle AMB(SSS)$ and, consequently, $m(\measuredangle(MDN)) = m(\measuredangle(EMD)) + m(\measuredangle(MED)) = m(\measuredangle(ABM)) + m(\measuredangle(BAM)) = m(\measuredangle(BMC))$.
Observe that $\triangle AMN$ is right angled isosceles with $AM = AN = a$, and, consequently, $(\measuredangle(MND)) = 135^\circ$ and $MN = \sqrt{2}a$.
From $\triangle MDN \equiv \triangle BMC(ASA)$ we have $BC = MN = \sqrt{2}a$ and $\frac{BC}{AM} = \sqrt{2}$.

Problem 69A. Proposed by Luca Vlad Andrei

Solve in $\mathbb{R}$ the equation $\left[\frac{2x+5}{6}\right] + \left[\frac{2x+7}{6}\right] = \left\{\frac{2x+3}{6}\right\} + \frac{8}{3}$.

Solution Problem 69A

Observations:

$$\frac{2x+3}{6} + \frac{1}{3} = \frac{2x+5}{6}$$
$$\frac{2x+3}{6} + \frac{2}{3} = \frac{2x+7}{6}$$
$$\left\{\frac{2x+3}{6}\right\} = \frac{2x+3}{6} - \left[\frac{2x+3}{6}\right]$$

The equation admits an equivalent form:

$$\left[\frac{2x+3}{6}\right] + \left[\frac{2x+3}{6} + \frac{1}{3}\right] + \left[\frac{2x+3}{6} + \frac{2}{3}\right] = \frac{2x+3}{6} + \frac{16}{6}$$

Applying Hermite Identity:

$$\left[\frac{2x+3}{6}\right] + \left[\frac{2x+3}{6} + \frac{1}{3}\right] + \left[\frac{2x+3}{6} + \frac{2}{3}\right] = \left[3\frac{2x+3}{6}\right]$$

we obtain

$$\left[3\frac{2x+3}{6}\right] = \frac{2x+19}{6}$$

For $\frac{2x+19}{6} = k$, $k \in \mathbb{Z} \Longrightarrow x = \frac{6k-19}{2}$ and

$$\left[\frac{6x-16}{2}\right] = k \Longrightarrow [3k-8] = k$$

But $k \in \mathbb{Z}$ and consequently $[3k-8] = 3k-8$. Finally, $3k-8 = k$ with $k = 4$ and $x = \frac{24-19}{2} = \frac{5}{2}$. The solution is $\frac{5}{2}$.

Problem 70A. Proposed by Vlad Nicolae Florescu

Let H be the orthocenter, I the incenter, and O the circumcenter of an acute triangle ABC. If $m(\measuredangle AHO) = m(\measuredangle AOH)$, show that $BHIC$ is an inscribable quadrilateral.

Solution Problem 70A

Let B', C' be the feet of altitudes from B and C respectively and R be the circumradius of $\triangle ABC$.

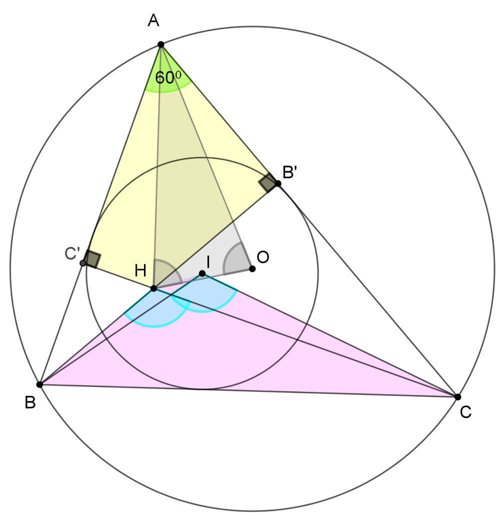


Figure 1:

- Showing that $AC'HB'$ is an inscribable quadrelateral
  $\measuredangle AHO = \measuredangle AOH \Rightarrow \triangle AHO$ is an isosceles triangle $\Rightarrow AH = OA = R$

$$\begin{cases} AH = R \\ AH = 2R\cos A \end{cases} \Rightarrow \measuredangle A = 60°$$

$$\begin{cases} BB' \perp AC \\ CC' \perp AB \end{cases} \Rightarrow AC'HB' \text{ inscribable quadrilateral}$$

- Showing that $BHIC$ is an inscribable quadrelateral
  $AC'HB'$ inscriable quadrilateral $\Rightarrow C'HB' = 180° - 60° = 120°$

  $$\begin{cases} C'HB' = 180° \\ C'HB' = BHC \end{cases} \Rightarrow \measuredangle BHC = 120°$$

  In $\triangle BHC$, $\measuredangle BHC = 180° - \frac{\measuredangle B}{2} - \frac{\measuredangle C}{2} = 180° - 60° = 120°$
  $\measuredangle BHC = \measuredangle BIC \Rightarrow BHIC$ is an inscribable quadrilateral.

Problem 71A. Proposed by Irina Daria Avram Popa

Find the values $a, b, c \in \mathbb{R}$ corresponding to the minimum of $E(a,b,c) = 3(a^2 + b^2 + c^2 + ab + ac + bc) + 4(a + b + c)$ and derive the minimum.

Solution Problem 71A

I propose to express $E(a,b,c)$ using the sum of the following squares.

$$\left(a + b + \frac{2}{3}\right)^2 = a^2 + b^2 + \frac{4}{9} + 2ab + \frac{4a}{3} + \frac{4b}{3} \quad (1)$$

$$\left(a + c + \frac{2}{3}\right)^2 = a^2 + c^2 + \frac{4}{9} + 2ac + \frac{4a}{3} + \frac{4c}{3} \quad (2)$$

$$\left(b + c + \frac{2}{3}\right)^2 = b^2 + c^2 + \frac{4}{9} + 2bc + \frac{4b}{3} + \frac{4c}{3} \quad (3)$$

So, (1)+(2)+(3) results:

$$\left(a + b + \frac{2}{3}\right)^2 + \left(a + c + \frac{2}{3}\right)^2 + \left(b + c + \frac{2}{3}\right)^2 = 2(a^2+b^2+c^2) + 2(ab+bc+ac) + \frac{8(a+b+c)}{3} + \frac{12}{9} =$$

$$\frac{6(a^2 + b^2 + c^2) + 6(ab + ac + bc) + 8(a + b + c)}{3} + \frac{4}{3}$$

$$S = \frac{2E(a,b,c)}{3} + \frac{4}{3}$$

We denote S the left side; being a sum of three squares we can conclude $S \geq 0$.
From

$$S \geq 0 \Rightarrow E(a,b,c) \geq -2$$

The minimum value of $E(a,b,c)$ is obtained for

$$a + b + \frac{2}{3} = a + c + \frac{2}{3} = b + c + \frac{2}{3} \Rightarrow a = b = c = -\frac{1}{3}$$

Problem 72A. Proposed by Octavian Tiberiu Bacain

Find the prime numbers p, q, and t such that $p^4 + q^4 + t^4 = 20151762$.

Solution Problem 72A

If p, q, and t primes, one of them must be even. Let us set $p = 2$; we obtain $q^4 + t^4 = 20151746$. For any number prime n different than 2 and 5, the last digit of n is 1. Thus, one number from q and t must be 5. Let us set $q = 5$; we obtain $t^4 = 20151121$ and $t = 67$. However, the problem is cyclic and the solutions are: $\{(2, 5, 67); (2, 67, 5); (5, 2, 67); (5, 67, 2); (67, 2, 5); (67, 5, 2)\}$.

Problem 73A. Proposed by Ana Boiangiu

Let ABC be a scalene triangle and let M be the midpoint of BC. Let the circumcircle of ΔAMB meet AC at D other than A. Similarly, let the circumcircle of ΔAMC meet AB at E other than A. Let N be the midpoint of DE. Prove that MN is parallel to the A-symmedian of ΔABC.

Solution Problem 73A

Without loss of generality, assume that $AC > AB$.

Note that by power of a point, $BD \cdot BA = BM \cdot BC = \frac{BC^2}{2}$ and $CE \cdot CA = CM \cdot CB = \frac{BC^2}{2}$. Therefore, $BD \cdot BA = CE \cdot CA$, or equivalently $\frac{BA}{CA} = \frac{CE}{BD}$.

Let (ADE) meet (ABC) a second time at K. It is well-known that K is the center of spiral similarity sending BD to CE. Therefore $\Delta KDB \sim \Delta KEC$ and thus $\frac{CE}{BD} = \frac{KC}{KB}$.

We can now conclude that $\frac{KB}{KC} = \frac{CA}{BA}$ and since $\measuredangle BKC = \measuredangle BAC$, we have $\Delta BKC \sim \Delta CAB$. This proves that $\measuredangle KCB = \measuredangle ABC$ or equivalently that $BAKC$ is an isosceles trapezoid.

Now since $\frac{ND}{NE} = \frac{MB}{MC} = \frac{1}{2}$, K is also the center of the spiral similarity sending MN to CE and therefore $\Delta KMN \sim \Delta KCE$. Consequently

$$\measuredangle KMN = \measuredangle KCE = \measuredangle KCA = \measuredangle KCB - \measuredangle ACB = \measuredangle ABC - \measuredangle ACB$$

since $BAKC$ is an isosceles trapezoid. But by symmetry $\measuredangle KMC = \measuredangle AMB$ and thus

$$\measuredangle NMC = \measuredangle KMN + \measuredangle KMC = \measuredangle ABC - \measuredangle ACB + \measuredangle AMB.$$

Let T be the foot of the A-symmedian of ΔABC. We want to compute angle $\measuredangle ATC$ and show that it is equal to angle $\measuredangle NMC$. Noting that $\measuredangle TAC = \measuredangle BAM$, we have

$$\measuredangle ATC = 180^\circ - \measuredangle TAC - \measuredangle TCA = 180^\circ - \measuredangle BAM - \measuredangle ACB$$

$$= 180^\circ - (180^\circ - \measuredangle AMB - \measuredangle ABM) - \measuredangle ACB = \measuredangle ABC - \measuredangle ACB + \measuredangle AMB.$$

Finally, $\measuredangle NMC = \measuredangle ATC$, which lets us conclude that $MN \parallel AT$, as desired.

Problem 74A. Proposed by Octavian Tiberiu Bacain

Prove that the following relationship is true for any real numbers a, b, c, and d:

$$(a+b+c)^2+(b+c+d)^2+(a+c+d)^2+(a+b+d)^2+116 \geq 2(13a+9b+17c+15d).$$

Solution Problem 74A

We will use the following notations: $x = (a+b+c)^2$, $y = (b+c+d)^2$, $z = (a+c+d)^2$ and $t = (a+b+d)^2$.
$13a+9b+17c+15d = 3(a+b+c)+5(b+c+d)+9(a+d+c)+(a+b+d) = 3x+5y+9z+t$
Using the notations, we have to prove that $x^2+y^2+z^2+t^2+116 \geq 2(3x+5y+9z+t)$ or

$$\frac{x^2+9}{2}+\frac{y^2+25}{2}+\frac{z^2+81}{2}+\frac{t^2+1}{2} \geq (3x+5y+9z+t)$$

But from AM-GM, $\frac{x^2+9}{2} \geq \sqrt{9x^2} = 3x$. Similarly, $\frac{y^2+25}{2} \geq 5y$, $\frac{z^2+81}{2} \geq 9z$ and $\frac{t^2+1}{2} \geq t$ Finally,

$$\frac{x^2+9}{2}+\frac{y^2+25}{2}+\frac{z^2+81}{2}+\frac{t^2+1}{2} \geq (3x+5y+9z+t)$$

is true.

Problem 75A. Proposed by Eliza Andreea Radu

Let a_1, a_2 . . .a_n be a sequence and $n \in \mathbb{N}$. Knowing that $a_1 = 121$ and

$$a_{n+1} = \frac{(n+3)\cdot a_n - 27}{n+6} \qquad \forall n \geq 1,$$

find all numbers n such that $a_n \in \mathbb{N}$.

Solution Problem 75A

We will derive first the values of the string of real numbers (b_n), where $b_n = a_n + 9$. We will use $a_n = b_n - 9$ and $a_{n+1} = b_{n+1} - 9$ for obtaining a recurrence for b_n.

$$b_{n+1} - 9 = \frac{(n+3)a_n - 27}{n+6} = \frac{(n+3)(b_n-9)-27}{n+6}$$
$$(b_{n+1}-9)(n+6) = (n+3)(b_n-9) - 27$$

$$nb_{n+1}+6b_{n+1}-9n-54 = nb_n-9n+3b_n-27-27 \iff b_{n+1}(n+6) = b_n(n+3) \iff b_{n+1} = \frac{n+3}{n+6}b_n$$

We can derive now the values of b_n:
$b_1 = 130$, $b_2 = \frac{4}{7}b_1$, $b_{n+1} = \frac{n+3}{n+6}b_n$.

$$b_1 \cdot b_2 \cdots b_{n+1} = 130 \cdot \frac{4}{7}b_1 \cdots \frac{n+3}{n+6}b_n$$

$$b_{n+1} = \frac{130 \cdot 4 \cdot 5 \cdot 6}{(n+3)(n+4)(n+5)}$$

$$b_n = \frac{130 \cdot 4 \cdot 5 \cdot 6}{(n+2)(n+3)(n+4)}$$

and

$$a_n = \frac{130 \cdot 4 \cdot 5 \cdot 6}{(n+3)(n+4)(n+5)} - 9$$

There are two values $n = 1$ and $n = 21$ such that b_n is natural. However, only $n = 1$ satisfies a_n being a natural number.

Problem 76A. Proposed by Vedaant Srivastava

Find all triples $(x, y, z) \in \mathbb{R}^3$ that satisfy the following system of equations:

$$\begin{cases} x^3 = -3x^2 - 11y + 26 \\ y^3 = 3y - 7z + 23 \\ z^3 = -9z^2 + 13x - 121 \end{cases}$$

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive integers $n_1, ..., n_k$ satisfying

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 56B. Proposed by Alexander Monteith-Pistor

A game is played with white and black pieces and a chessboard (8 by 8). There is an unlimited number of identical black pieces and identical white pieces. To obtain a starting position, any number of black pieces are placed on one half of the board and any number of white pieces are placed on the other half (at most one piece per square). A piece is called matched if its color is the same of the square it is on. If a piece is not matched then it is mismatched. How many starting positions satisfy the following condition

$$\#\text{ of matched pieces} - \#\text{ of mismatched pieces} = 16$$

(your answer should be a binomial coefficient)

Problem 62B. Proposed by Eliza Andreea Radu

If $a_1, a_2, \ldots, a_{2021} \in \mathbb{R}_+$ such that $\sum_{i=1}^{2021} a_i > 2021$, prove that

$$a_1^{2^{2021}} \cdot 1 \cdot 2 + a_2^{2^{2021}} \cdot 2 \cdot 3 + \ldots + a_{2021}^{2^{2021}} \cdot 2021 \cdot 2022 > 4086462.$$

Problem 63B. Proposed by Alexandru Benescu

Prove that

$$\frac{\sqrt{1^6+1}}{1^2} + \frac{\sqrt{2^6+1}}{2^2} + \frac{\sqrt{3^6+1}}{3^2} + \ldots + \frac{\sqrt{2020^6+1}}{2020^2} > \frac{\sqrt{(2021^2 \cdot 1010)^2 + 2020^2}}{2021}.$$

Solution Problem 63B

Taking a look at a general case of the above square root, we have

$$\frac{\sqrt{n^6+1}}{n^2} = \sqrt{\frac{n^6+1}{n^4}} = \sqrt{\frac{n^6}{n^4} + \frac{1}{n^4}} = \sqrt{\left(\frac{1}{n}\right)^4 + n^2}.$$

Let $S = \frac{\sqrt{1^6+1}}{1^2} + \frac{\sqrt{2^6+1}}{2^2} + \frac{\sqrt{3^6+1}}{3^2} + \ldots + \frac{\sqrt{2020^6+1}}{2020^2}$.

We can rewrite it as:

$$S = \sqrt{1^4 + 1^2} + \sqrt{\left(\frac{1}{2}\right)^4 + 2^2} + \cdots + \sqrt{\left(\frac{1}{2020}\right)^4 + 2020^2}.$$

We define the following sequences of numbers:

$$a_1 = 1^2, a_2 = \left(\frac{1}{2}\right)^2, \cdots, a_{2020} = \left(\frac{1}{2020}\right)^2$$

and

$$b_1 = 1, b_2 = 2, \cdots, b_{2020} = 2020.$$

We can apply the following inequality (*Minkowski*):

$$\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+\cdots+\sqrt{a_{2020}^2+b_{2020}^2} \geq \sqrt{(a_1+a_2+\cdots\ a_{2020})^2+(b_1+b_2+\cdots\ b_{2020})^2}.$$

Proof of the Above Inequality: Added by AEs

We have that

$$\text{RHS}^2 = \sum_{i=1}^{2020}(a_i^2+b_i^2) + 2\sum_{i=1}^{2019}\sum_{j=i+1}^{2020}(a_ia_j+b_ib_j),$$

$$\text{LHS}^2 = \sum_{i=1}^{2020}(a_i^2+b_i^2) + 2\sum_{i=1}^{2019}\sum_{j=i+1}^{2020}\sqrt{(a_i^2+b_i^2)(a_j^2+b_j^2)}.$$

We know via AM-GM that

$$a_i^2b_j^2 + a_j^2b_i^2 \geq 2a_ia_jb_ib_j.$$

Thus,

$$a_i^2a_j^2 + a_i^2b_j^2 + a_j^2b_i^2 + b_i^2b_j^2 \geq (a_ia_j+b_ib_j)^2,$$

which means that

$$(a_i^2+b_i^2)(a_j^2+b_j^2) \geq (a_ia_j+b_ib_j)^2.$$

Therefore,

$$2\sum_{i=1}^{2019}\sum_{j=i+1}^{2020}\sqrt{(a_i^2+b_i^2)(a_j^2+b_j^2)} \geq 2\sum_{i=1}^{2019}\sum_{j=i+1}^{2020}(a_ia_j+b_ib_j),$$

giving us that $\text{LHS}^2 \geq \text{RHS}^2 \Rightarrow \text{LHS} \geq \text{RHS}$. The inequality is thus true.

We obtain:

$$\begin{aligned}
S &= \sqrt{1^4+1^2} + \sqrt{\left(\frac{1}{2}\right)^4 + 2^2} + \cdots + \sqrt{\left(\frac{1}{2020}\right)^4 + 2020^2} \\
&\geq \sqrt{\left[1^2 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2020}\right)^2\right]^2 + (1+2+\cdots+2020)^2} \\
&= \sqrt{\left(\frac{1}{1\cdot 1} + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2020\cdot 2020}\right)^2 + \left(\frac{2020\cdot 2021}{2}\right)^2} \\
&= \sqrt{\left(\frac{1}{1\cdot 1} + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2020\cdot 2020}\right)^2 + (1010\cdot 2021)^2} \\
&> \sqrt{\left(1 - \frac{1}{2021}\right)^2 + 1010^2\cdot 2021^2} \\
&= \sqrt{\left(\frac{2020}{2021}\right)^2 + 1010^2\cdot 2021^2} \\
&= \sqrt{\frac{2020^2 + 1010^2\cdot 2021^4}{2021^2}} \\
&= \frac{\sqrt{(2021^2\cdot 1010)^2 + 2020^2}}{2021}.
\end{aligned}$$

Problem 67B. Proposed by Stefan-Ionel Dumitrescu

Consider a cube $ABCDA'B'C'D'$. Point X lies on face $ADD'A'$. Point Y lies on face $ABB'A'$. Point W is a randomly chosen point on the edges, faces, or interior of the cube. If Z is the midpoint of XY, find the probability that W is the midpoint of an AZ.

Problem 69B. Proposed by Frederick Pu

Consider the following game: The first turn consists of placing one card on the table. Each turn after the first, insert a new card in a random place in the pile. If the new card you inserted is not at the top of the deck you win. Otherwise, keep playing.
What is the probability that you win eventually?

Solution Problem 69B

Let

$$L_n = \sum_{i=1}^{n} \frac{i}{(i+1)!}$$

You win on turn $i+1$ if every turn before turn $i+1$ you inserted a card on the top of the deck and you don't insert a card on the top of the deck on turn $i+1$. Thus, the probability that you win on turn $i+1$ is $\frac{1}{i!} \cdot \frac{i}{i+1} = \frac{i}{(i+1)!}$.

Via rule of sum, we can conclude that L_n is the probability that you win in $n+1$ turns.
So, $1 - L_n$ is the probability you don't win in n turns. This occurs if you if you inserted the card on the top of the deck every turn. Thus, $1 - L_n = \frac{1}{(n+1)!}$. Call this fact (A).

It follows that the probability you never win is.

$$\lim_{n\to\infty} [1 - (1 - L_n)] = 1 - \lim_{n\to\infty} (1 - L_n) = 1 - \lim_{n\to\infty} \frac{1}{(n+1)!} = 1$$

Thus, you are garanteed to win.

Fact (A) can also be shown via induction.

Base case: $n = 1$

$$L_1 = \sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{2!} = \frac{1}{2}$$

$1 - L_1 = 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{(1+1)!}$
Suppose $1 - L_n = \frac{1}{(n+1)!}$:

$$\begin{aligned} 1 - L_{n+1} &= 1 - \sum_{i=1}^{n+1} \frac{i}{(i+1)!} \\ &= 1 - (\sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!}) \\ &= 1 - \sum_{i=1}^{n} \frac{i}{(i+1)!} - \frac{n+1}{(n+2)!} \\ &= \frac{1}{(n+1)!} - \frac{n+1}{(n+2)!} \\ &= \frac{n+2}{(n+2)!} - \frac{n+1}{(n+2)!} \\ &= \frac{1}{(n+2)!} \end{aligned} \tag{4}$$

By induction, $1 - L_n = \frac{1}{(n+1)!}$.

Problem 70B. Proposed by Daisy Sheng

Triangle ABC is obtuse where $\measuredangle C > 90°$. Show that

$$4r^2 \leq \frac{a^2b^2c^2}{(a+b+c)^2(c^2-a^2-b^2)},$$

where r is the inradius of $\triangle ABC$ and a, b, c represent the length of the sides opposite to $\measuredangle A, \measuredangle B, \measuredangle C$, respectively.

Problem 71B. Proposed by Andrew Dong

Let $n \geq 3$ be an integer and let $T_1, T_2, \ldots, T_{n-2}$ be the $n-2$ triangles of a triangulation $\mathcal{T}$ of a convex n-gon. Construct an undirected graph $G = (V, E)$ with $V = [n]$ and $(u, v) \in E$ if and only if T_u and T_v share an edge in $\mathcal{T}$. Show that G is a tree.

Solution Problem 71B

We use induction. The result clearly holds for $n = 3$.

Suppose the result holds for $n = k - 1$ and consider a triangulation $\mathcal{U}$ of a k-gon P. Since there are $k-2$ triangles and k sides, by the Pigeonhole Principle, some triangle T_L of $\mathcal{U}$ must have two sides coinciding with the sides of the k-gon.

It is easy to check that, if a triangle has two sides along the k-gon, these two sides must be consecutive sides of the k-gon. Hence, in G, node L is a leaf. We finish by applying the inductive hypothesis on $n = k - 1$ to the $(k-1)$-gon formed by chopping off T_L from P, which is also convex.

Problem 72B. Proposed by Daisy Sheng

For $x \in \mathbb{R}$, the solution to

$$5^{2x^4-10x^2+9} + 4 \cdot 45^{x^4-5x^2+4} \geq 3^{4x^4-20x^2+18}$$

is $x \in [-b, -a] \cup [a, b]$, where $a, b \in \mathbb{Z}^+$. Find a and b.

Solution Problem 71B

Let $y = x^4 - 5x^2 + 4$.

Our inequality becomes

$$5^{2y+1} + 4 \cdot 45^y \geq 3^{4y+2}.$$

Since $45 = 5 \cdot 9$, we have

$$5^{2y+1} + 4 \cdot 5^y \cdot 3^{2y} \geq 3^{4y+2}.$$

Setting $m = 5^y$ and $n = 3^{2y}$, we get

$$5m^2 + 4mn \geq 9n^2,$$

which rearranges and factors to become

$$(5m + 9n)(m - n) \geq 0.$$

Since $m, n > 0$, we know that the inequality will be satisfied as long as $m - n \geq 0$. So we are looking for y such that

$$5^y - 3^{2y} \geq 0 \Rightarrow 5^y - 9^y \geq 0.$$

This is only true when $y \leq 0$. Therefore,

$$x^4 - 5x^2 + 4 = (x + 2)(x + 1)(x - 1)(x - 2) \leq 0.$$

Creating a table, we get

	-∞	-2		-1		1		2	∞
x+2	− −	0	+ + +	+	+ +	+	+	+	
x+1	− −	−	− −	0	+ +	+	+	+	
x-1	− −	−	− −	−	− −	0	+	+	
x-2	− −	−	− −	−	− −	−	−	0	+
Product	+ +	0	− −	0	+ +	0	− −	0	+

Thus, $x \in [-2, -1] \cup [1, 2]$, meaning that $a = 1$ and $b = 2$.

Problem 73B. Proposed by Andrew Dong

Consider $n \geq 3$ distinct points in the plane. Show that there exists a triple of three distinct points (A, B, C) such that $\measuredangle ABC \leq \pi/(n - 2)$.

Solution Problem 73B

Let P be a point on the convex hull of the n points, and consider the points to be in the Cartesian plane. By rotating and shifting, we may assume that $P = (0, 0)$ and that the remaining $n - 1$ points have non-negative y-coordinate.

Label the remaining points $P_1, P_2, \ldots, P_{n-1}$ in increasing order of polar angle. Note that

$$\sum_{i=1}^{n-2} \measuredangle P_i P P_{i+1} = \measuredangle P_1 P P_{n-1} \leq \pi$$

so the smallest of the $n-2$ terms of the form $\measuredangle P_i P P_{i+1}$ is at most $\pi/(n-2)$. For this i that yields the minimum $\measuredangle P_i P P_{i+1}$, we can take $(A, B, C) = (P_i, P, P_{i+1})$.

Problem 74B. Proposed by Alexander Monteith-Pistor

Prove that, for all positive integers n which are relatively prime to 2021, 2021 divides

$$n^{(n^{281}-281)(n^{280}-280)\dots(n^2-2)(n-1)} - 1$$

Solution Problem 74B

Let $2021 = 43 \cdot 47$. By Fermat's Little Theorem, it suffices to show that for all n not divisible by 43 or 47, 42 and 46 divide

$$(n^{275} - 275)(n^{274} - 274)\dots(n^2 - 2)(n - 1)$$

One can verify that 2 divides $(n^2 - 2)(n - 1)$ and 3 divides $(n^4 - 1)(n^3 - 3)$. Further, 7 divides $(n^{36} - 36)(n^7 - 7)$ since n^{36} is congruent to 1 modulo 7 for n not divisible by 7. Thus, 42 divides the expression. It remains to show that 23 divides

$$(n^{275} - 275)(n^{274} - 274)\dots(n^2 - 2)(n - 1)$$

If n is divisible by 23 then this clearly holds since 23 divides $n^{23}-23$. Otherwise, by Fermat's Little Theorem, $n^{22} \equiv 1 \mod 23$. Recall that $n^2 \equiv 1 \mod 23$ implies 23 divides $n^2 - 1 = (n - 1)(n + 1)$ and so $n \equiv \pm 1 \mod 23$. Therefore, $n^{11} \equiv \pm 1 \mod 23$. Finally, we have

$$(n^{275} - 275)(n^{231} - 231) \equiv (n^{11} + 1)(n^{11} - 1) \mod 23$$

which implies the desired result.

Problem 75B. Proposed by Max Jiang

Does every point in the unit circle lie on the polar curve $r = \cos(\theta^2)$?

Solution Problem 75B

The answer is no. Consider the point $(-1, 0)$ (in rectangular coordinates), whose polar coordinates are in the form $(r, \theta) = (1, (2k + 1)\pi), k \in \mathbb{Z}$ or $(r, \theta) = (-1, 2k\pi), k \in \mathbb{Z}$. If such a point were to lie on the curve, we would have

$$r = \cos(\theta^2)$$
$$\implies 1 = \cos(((2k + 1)\pi)^2)$$
$$\implies \cos(((2k + 1)^2\pi)\pi) = 1$$

or

$$r = \cos(\theta^2)$$
$$\implies -1 = \cos((2k\pi)^2)$$
$$\implies \cos((4k^2\pi)\pi) = -1.$$

We see that $x = n\pi, n \in \mathbb{Z}$ when $\cos(x) = \pm 1$. Since $(2k+1)^2 \in \mathbb{Z} \setminus \{0\}$ and π is irrational, $(2k+1)^2\pi$ is never an integer. $4k^2\pi$ is only an integer when $k = 0$, which gives $(r, \theta) = (1, 0)$, which is also the point $(1, 0) \neq (-1, 0)$ in rectangular coordinates. Thus, $(-1, 0)$ does not lie on the curve $r = \cos(\theta^2)$.

Problem 76B. Proposed by Alexander Monteith-Pistor

Let $ABCD$ be a quadrilateral with $\measuredangle ABC = 90°$. Points E and F are on AD and BC respectively such that AB is parallel to EF. Further, AC, BD and EF intersect at O. Given that $BF = 4$, $AB = 9$, $AE = 5$ and $CD = 20$, find a polynomial $p(x)$ such that one of its roots is at $x = \frac{DO}{OB}$.

Problem 77B. Proposed by Andy Kim

(i) Evaluate

$$\binom{n}{0} - 2\binom{n}{1} + \cdots \pm 2^n\binom{n}{n} = \sum_{i=0}^{n}(-1)^i 2^i \binom{n}{i}$$

for $n \in \mathbb{Z}_+$.
(ii) Prove that

$$\sum_{i=0}^{n}(-1)^{n-i} i^n \binom{n}{i} = n!$$

for all $n \in \mathbb{Z}_+$.

Problem 78B. Proposed by Ciurea Pavel

Given the positive real numbers x, y, and z, prove that

$$2(\sum_{cyc} x)\sqrt{\sum_{cyc}\sqrt{x^2+y^2+z^2}} \geq$$

$$\geq \sum_{cyc}\sqrt{3(x+y)(x+z)(\sqrt{x^2+y^2+xy}+\sqrt{x^2+z^2+xy}-\sqrt{y^2+z^2+yz}}).$$