Number 2 - October 2020 Problems

November 21, 2020

Problem 5A. Proposed by Alexander Monteith-Pistor

Let $\triangle ABC$ be an isosceles right triangle with right angle at A and incenter I. The point P is randomly chosen inside $\triangle ABC$. Find the probability that $\triangle PAI$ is an acute triangle (all of its angles are acute).

Solution:

$\triangle ABC$ is isosceles therefore I lies on the altitude of $\triangle ABC$ passing through A. Let B_1, C_1 be on AB, AC respectively such that I lies on B_1C_1 and B_1C_1 is parallel to BC. Then $\measuredangle AIP$ is acute if and only if I lies inside triangle AB_1C_1.

Let AB and AC be tangent to the incircle of $\triangle ABC$ at B_2 and C_2 respectively. Since $B_2I = C_2I$, AB_2IC_2 is a square. Thus, $\measuredangle API$ is acute if and only if I lies outside of $\mathcal{C}$, which we define as the circle with diameter AI. Note that $\measuredangle PAI$ will be acute regardless of the position of P.

It follows that $\triangle API$ is acute if and only if P is inside $\triangle AB_1C_1$ and outside $\mathcal{C}$. Let $\triangle ABC$ have inradius r and area k. Note $k = (\sqrt{2}+1)^2r^2$ ($\triangle ABC$ is similar to $\triangle AB_2C_2$). Since AB_2IC_2 is a square with side length r, $AI = \sqrt{2}r$. Further, the altitude of $\triangle ABC$ passing through A (and thus through I) has length $(\sqrt{2}+1)r$. Then, using similarity between $\triangle ABC$, $\triangle AB_1C_1$, $\triangle AB_2C_2$, the area of $B_1B_2C_2C_1$ is

$$\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)^2 k - \left(\frac{\sqrt{2}}{2(\sqrt{2}+1)}\right)^2 k = \frac{3}{2(\sqrt{2}+1)^2}k$$

Note that $\triangle AB_2C_2$ is contained within its circumcircle, $\mathcal{C}$. Therefore, the region contained in $\triangle AB_1C_1$ but not in $\mathcal{C}$ has area

$$\frac{3}{2(\sqrt{2}+1)^2}k - \frac{1}{2}\left(\frac{\sqrt{2}}{2}r\right)^2 \pi$$

Hence, the probability that $\triangle API$ is acute is

$$\frac{1}{k}\left(\frac{3}{2(\sqrt{2}+1)^2}k - \frac{1}{4}r^2\pi\right) = \boxed{\frac{6-\pi}{4(\sqrt{2}+1)^2}}$$

Problem 11A. Proposed by Luca Tu

Call a number C unlucky if there exists a multiset $S \subseteq \mathbb{N}^*$ such that the sum of all the elements in S is 169, and the product of all the elements in S is C. Find the largest unlucky number. (A multiset is a set that can have duplicate elements.)

Solution:

This could seem intimidating at first, but what the problem is basically asking is for the solver to find a set S such that all its elements sum to 169, and the product is maximized. Now, 169 is quite a big number. We first consider the same problem with 24 (instead of 169) and observe some of the possibilities:

$$2^{12} = 4096, 2 \cdot 12 = 24$$
$$3^8 = 6561, 3 \cdot 8 = 24$$
$$4^6 = 4096$$
$$5^5 = 3125 \text{ (close enough)}$$
$$6^4 = 1296$$
$$7^3 = 343$$
$$7^4 = 2401$$
$$8^3 = 512$$

(1 can't be a possibility as it just wastes a unit) These are the "monotonic" sets, with one number for every element. It seems like 3 is the "best" one, resulting in the biggest product. Now we prove $3^a > a^3$, using induction.

1. Base Case.

$$3^4 = 81 > 64 = 4^3$$

2. Induction phase: Assuming that $3 \cdot (a-1) > (a-1)^3$ for some natural number $a > 4$.

We want to prove that $3^a > a^3$. Multiply by 3 on both sides of the first inequality:

$$3^a > 3 \cdot (a-1)^3$$

And now, we want to prove that $3 \cdot (a-1)^3 > a^3$.

$$\begin{aligned}
\text{LHS} - \text{RHS} &= 3 \cdot (a-1)^3 - a^3 \\
&= 3 \cdot (a^3 - 3a^2 + 3a - 1) - a^3 \\
&= 3a^3 - a^3 + 3 \cdot (-3a^2 + 3a - 1) \\
&= 2a^3 - 9a^2 + 9a - 3 \\
&= (2a^3 - 12a^2 + 18a) + 3a^2 - 9a - 3 \\
&= 2a(a-3)^2 + 3(a^2 - 3a - 1)
\end{aligned}$$

The first term is clearly positive. We will find the x-intercepts of the second term:

$$\frac{3 \pm \sqrt{\{9+4}}{2} = \frac{3+\sqrt{13}}{2} < \frac{3+4}{2} < 3.5$$

Therefore, $a^2 - 3a - 1$ is never negative for $a > 3.5$, and the second term is also positive. Hence LHS > RHS, and the inductive step is complete.

Now, we want to prove that monotonic sets of 3, with one or two 2's, always maximize the products of the elements (given that the sum of the elements is fixed).

Consider a set multiset S. Let $x \in S$ with $x > 3$. Then, using the above result: if $x = 3k$, replace x with k 3's to get a better set S'; if $x = 3k+1$, replace x with two 2's and $k-1$ 3's to obtain a better set S'; if $x = 3k+2$, replace x with k 3's and one 2 two get a better set S'. Finally, observe that including 1's is counterproductive (since they contribute nothing to the product) and $3^2 > 2^3$. Thus, the best sets contain only 3's (and maybe one or two 2's).

Now we know that the optimal set would constitute of only 3's (and maybe one or two 2's). Let put it into context!

$$169 = 3 \cdot 56 + 1 = 3 \cdot 55 + 2 \cdot 2$$

Therefore, the biggest unlucky number is $3^{55} \cdot 2^2$.

Problem 12A. Proposed by Luca Tu

In the following calculation, each letter stands for a different digit (e.g. if $X = 0$ then $Y \neq 0$)

$$\begin{array}{ccccccc}
 & C & L & O & V & E & R \\
+ & C & R & O & C & U & S \\
\hline
 & V & I & O & L & E & T
\end{array}$$

Decode the equation (find the numbers CLOVER, CROCUS and VIOLET). (Note O in the equation is the letter O and not the digit 0)

Solution:

Throughout this proof, things in square brackets will be possibilities (they may or may not be true). First of all, we see that

$$O + O + [\text{potential carry}] = O + [10].$$

This means that either O = 0 and there's no carry, or O = 9 and there is a carry. Also,

$$E + U + [\text{carry}] = E + [10]$$

so same thing, U = 0 or U = 9.

If O = 9, then U = 0. However,

$$C + C + [\text{carry}] = V, \text{ and } C + V = L + 10 \text{ (Can't have carry as } U = 0).$$

since $L \neq 0$, C = 4. Then V = 8, L = 2. Since R + S = T, and we only have 1, 3, 5, 6, 7 not assigned, R = 1, 5, or 6. However, I = R + 3 implies R = 3 (we only have 1, 3, 5, 6, 7 not assigned). This is clearly a contradiction.

Therefore, O = 0, U = 9.
Then C + C + (carry) = V, and V + C + 1 = L (as U = 9). Hence, C = 1 or 2.

Case 1: C = 1. Either V = 2, L = 4 or V = 3, L = 5.

If V = 2, L = 4. Then R + S = 10 + T, 4 + R = I, we have 3, 5, 6, 7, 8 left. Hence R = 3, I = 7, but the other equation fails.

So V = 3, L = 5. Then R + S = 10 + T, 5 + R = I + 10, we have 2, 4, 6, 7, 8 left. Hence R = 7, I = 2, but the other equation fails.

Case 2: C = 2. Rewrite the equation for clarity:

$$\begin{array}{ccccccc} & 2 & L & 0 & V & E & R \\ + & 2 & R & 0 & C & 9 & S \\ \hline & V & I & 0 & L & E & T \end{array}$$

we have either V = 4 or V = 5.

If V = 4: Then L = 7, R + S = 10 + T, 7 + R = I, and we have 1, 3, 5, 6, 8 left. Hence R = 1, I = 8, but the other equation fails.

So V = 5. Then L = 8, R + S = 10 + T, 8 + R = 10 + I, and we have 1, 3, 4, 6, 7 left. Either R = 3, I = 1, or R = 6, I = 4. If R = 3, 3 + S = 10 + T which fails (we have 4, 6, 7 left).

Therefore, R = 6, I = 4 and 6 + S = 10 + T, we have 1, 3, 7 left. Luckily, S = 7, T = 3 works. Then E = 1 by elimination, and we are done.

The equation is then $280516 + 260297 = 540813$. Rewriting the whole thing:

$$\begin{array}{ccccccc} & 2 & 8 & 0 & 5 & 1 & 6 \\ + & 2 & 6 & 0 & 2 & 9 & 7 \\ \hline & 5 & 4 & 0 & 8 & 1 & 3 \end{array}$$

And it does indeed work.

Problem 13A. Proposed by Vedaant Srivastava

Let $A_1B_1C_1$ be a triangle with $A_1B_1 = 13$, $B_1C_1 = 14$, $C_1A_1 = 15$. Let $\Gamma_1 : (O_1, r_1)$ be the incircle of $A_1B_1C_1$. Construct B_2, C_2 on sides A_1B_1 and A_1C_1 respectively such that $B_2C_2 \parallel B_1C_1$ and B_2C_2 is tangent to Γ_1.

Now let $\Gamma_2 : (O_2, r_2)$ be the incircle of $\triangle AB_2C_2$. Construct B_3, C_3 on sides AB_2, AC_2 respectively such that $B_3C_3 \parallel B_2C_2$ and B_3C_3 is tangent to Γ_2.

Continue this process, shading in the circles $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \ldots$. What is the total area of the shaded region?
Solution:
Let s be the semiperimeter of $A_1B_1C_1$. We denote the area of a figure with square brackets.

First determine the area of Γ_1. We have that

$$r_1 = [\triangle A_1B_1C_1]/s = 84/21 = 4$$

Thus the area of Γ_1 is 16π.

Now consider the excircle $\Gamma_0 : (O_0, r_0)$ of $\triangle A_1B_1C_1$ tangent to side B_1C_1.
By homothety from point A_1, observe that

$$\frac{r_{i+1}}{r_i} = \frac{r_1}{r_0} = \frac{s - B_1C_1}{s} = \frac{21-14}{21} = \frac{1}{3}$$

Where the second equality follows from the well-known fact that $\frac{r}{r_A} = \frac{s-a}{s}$.

This implies that

$$\frac{[\Gamma_{i+1}]}{[\Gamma_i]} = \left(\frac{r_{i+1}}{r_i}\right)^2 = \frac{1}{9}$$

Thus we have

$$\sum_{i=1}^{\infty}[\Gamma_i] = 16\pi\sum_{i=0}^{\infty}\left(\frac{1}{9}\right)^i = \frac{16\pi}{1-\frac{1}{9}} = \boxed{18\pi}$$

Where the second equality follows from the formula for an infinite geometric series.

Problem 14A. Proposed by Vedaant Srivastava

Consider an acute triangle ABC with orthocenter H. Let H_A be the reflection of H over BC. Let E and F be the projections of H_A onto AB and AC respectively. Prove that EF bisects HH_A.

Solution:

In order to bypass configuration issues, we use directed angles in the following proof, where angle measures are measured counterclockwise and taken $\pmod{180^\circ}$.

Proof. Let D be the projection of H_A onto BC.
We have that

$$\angle BH_AC = \angle CHB = \angle CHD + \angle DHB = \angle ABC + \angle BCA = -\angle CBA = \angle BAC$$

So H_A lies on the circumcircle of $\triangle ABC$.

Now observe that $\angle H_AEB = \angle H_ADB = 90^\circ$ so H_A, E, B, D are concyclic.
Similarly, $\angle H_AFC = \angle H_ADC = 90^\circ$ so H_A, F, C, D are concyclic.

This implies that

$$\angle H_ADE = \angle H_ABE = \angle H_ABA = \angle H_ACA = \angle H_ADF$$

So by definition, D, E, F collinear.

Observing that D is the midpoint of HH_A, we have that EF bisects HH_A at D, as desired. □

Problem 15A. Proposed by Alexander Monteith-Pistor

Let $x_0 = 0$. For $n \geq 1$: if x_{n-1} is even, x_n is randomly chosen from the multiples of 3 between $x_{n-1} - 10$ and $x_{n-1} + 10$ (inclusive); if x_{n-1} is odd, x_n is randomly chosen from the multiples of 4 between $x_{n-1} - 10$ and $x_{n-1} + 10$ (inclusive). Find the probability that x_5 is even.

Problem 16A. Proposed by Nicholas Sullivan

The soccer ball, otherwise known as the truncated icosahedron, is composed of 12 pentagonal faces and 20 hexagonal faces, such that one pentagon and two hexagons meet at each vertex. If each vertex is labelled with a distinct integer between 1 and 70, inclusive, show that there must be at least one edge whose vertices are not relatively prime.

Solution:

Since each vertex borders exactly one pentagon of the truncated icosahedron, and since there are 12 pentagonal faces, then there must be 60 vertices in total. If each is labelled with a distinct integer between 1 and 70, then there can be at most 35 odd vertices, and thus at least 25 even vertices. Each of these 25 even vertices must belong to at exactly one pentagonal face. By the pigeonhole principle, since there are 12 pentagonal faces, and 25 even vertices to distribute between them, there must be at least one pentagonal face with 3 even vertices. On such a pentagonal face with 3 even vertices, there must be at least one edge with two even vertices. Thus, there must exist an edge whose vertices are not relatively prime.

Problem 17A. Proposed by Nikola Milijevic

Express the sum of A and B in simplest form if

$$A = \sqrt{7 + 2\sqrt{6}} - \sqrt{7 - 2\sqrt{6}}$$

and

$$B = 1 + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A + \dots}}}$$

Solution:
If we square A, we get

$$A^2 = \sqrt{7 + 2\sqrt{6}}^2 - 2\sqrt{7 + 2\sqrt{6}}\sqrt{7 - 2\sqrt{6}} + \sqrt{7 - 2\sqrt{6}}^2$$

$$= 7 + 2\sqrt{6} - 2\sqrt{25} + 7 - 2\sqrt{6} = 4$$

As A is positive, it must mean $A = 2$. Notice that $B = 1 + \frac{1}{B+A-1}$, so

$$B^2 + AB - B = B + A - 1 + 1$$

$$B^2 + (A - 2)B - A = 0$$

$$B^2 - 2 = 0$$

$$B = \sqrt{2}$$

So $A + B = 2 + \sqrt{2}$

Problem 18A. Proposed by Andy Kim

Find the probability that a randomly chosen 6-digit palindrome with no digits equal to 0 is divisible by 13.
Solution:
We represent the six-digit palindrome as $\overline{abccba}$. Then,

$$\overline{abccba} = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + c \cdot 10^2 + b \cdot 10^1 + a$$

Noting that $10 \equiv -3 \pmod{13}$, we have

$$\begin{aligned}
& a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + c \cdot 10^2 + b \cdot 10^1 + a && \pmod{13} \\
\equiv\ & a \cdot (-3)^5 + b \cdot (-3)^4 + c \cdot (-3)^3 + c \cdot (-3)^2 + b \cdot (-3)^1 + a && \pmod{13} \\
\equiv\ & -243a + 81b - 27c + 9c - 3b + a && \pmod{13} \\
\equiv\ & -242a + 78b - 18c && \pmod{13} \\
\equiv\ & -8a + 8c && \pmod{13} \\
\equiv\ & 8(c - a) && \pmod{13}
\end{aligned}$$

So, since 13 is prime, $\overline{abccba}$ is divisible by 13 if and only if $c - a$ is divisible by 13. This can only happen if $a = c$, so the probability of the palindrome being divisible by 13 is the probability that $a = c$, which is $\frac{1}{9}$.

Problem 19A. Proposed by Luca Tu

Solve for $x \in \mathbb{R}$: $x^3 + 10x^2 + 25x + 6 = 0$ without using Cardano's formula or its equivalent. Show your work.

Solution:

If you take a close look at the equation's coefficients, we see that we can rewrite it as:

$$xm^2 + (2x^2 + 1)m + x^3 + 1 = 0 \text{ where } m = 5$$

Therefore, we can find a cleaner equation for x if we take x as the coefficients and 5 as the variable, m.

$$\begin{aligned}
m &= \frac{-(2x^2 + 1) \pm \sqrt{(2x^2 + 1)^2 - 4(x^3 + 1)x}}{2x} \\
&= \frac{-(2x^2 + 1) \pm \sqrt{4x^4 + 4x^2 + 1 - 4x^4 - 4x}}{2x} \\
&= \frac{-(2x^2 + 1) \pm \sqrt{4x^2 - 4x + 1}}{2x} \\
&= \frac{-(2x^2 + 1) \pm (2x - 1)}{2x} = 5
\end{aligned}$$

Therefore,

$$-2x^2 - 1 \pm (2x - 1) = 10x$$

Case 1: $-2x^2 - 1 + (2x - 1) = 10x$

$$-2x^2 - 2 - 8x = 0$$

$$x^2 + 4x + 1 = 0$$

which gives the solutions $x = -2 - \sqrt{3}$ and $x = \sqrt{3} - 2$.

Case 2: $-2x^2 - 1 - (2x - 1) = 10x$

$$-2x^2 = 12x$$

$$-2x = 12$$
$$x = -6$$

Hence the solutions for the original are $x = -6$, $x = -2 - \sqrt{3}$, and $x = \sqrt{3} - 2$.

Problem 20A. Proposed by DC

In a cyclic quadrilateral ABCD, P is the intersection of the diagonals and E, M, G and L are the midpoints on sides AB, BC, CD and DA. By projecting the medians PE, PM, PG and PL on sides AB, BC, CD and DA, four segments are obtained: EF, MN, GH and LI. Prove that

$$AB \times FE + CD \times HG = AD \times IL + CB \times MN.$$

Problem 8B. Proposed by Andy Kim

Let a, b, and c be positive real numbers. Given that $abc = 1$, prove that

$$\frac{a^3-1}{b^2c^2} + \frac{b^3-1}{c^2a^2} + \frac{c^3-1}{a^2b^2} \geq 0$$

Solution:

$$\frac{a^3-1}{b^2c^2} + \frac{b^3-1}{c^2a^2} + \frac{c^3-1}{a^2b^2} \geq 0$$
$$(a^5 - a^2) + (b^5 - c^2) + (c^5 - c^2) \geq 0$$
$$a^5 + b^5 + c^5 \geq a^2 + b^2 + c^2$$

Then, we homogenize the inequality by multiplying the right hand side by $abc = 1$.

$$a^5 + b^5 + c^5 \geq abc(a^2 + b^2 + c^2)$$
$$a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab$$

Now, the condition $abc = 1$ can be ignored. From AM-GM, we have

$$\frac{a^5 + a^5 + a^5 + b^5 + c^5}{5} \geq \sqrt[5]{a^5a^5a^5b^5c^5}$$
$$\frac{3a^5 + b^5 + c^5}{5} \geq a^3bc \tag{1}$$

Similarly, we also have

$$\frac{3b^5 + c^5 + a^5}{5} \geq b^3ca \quad (2)$$
$$\frac{3c^5 + a^5 + b^5}{5} \geq c^3ab \quad (3)$$

Summing (1), (2), and (3), we obtain the desired inequality.

Problem 11B. Proposed by Max Jiang

Find all complex roots of the polynomial

$$x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1 = 0.$$

Solution:
The important thing to note about this polynomial is that its coefficients are "symmetrical" about the x^4 term. Let us group the terms by their coefficient to get

$$(x^8 + 1) - 4(x^7 + x) + 10(x^6 + x^2) - 16(x^5 + x^3) + 19x^4 = 0.$$

Clearly $x = 0$ is not a root, so we can safety divide by x^4 to get

$$(x^4 + \frac{1}{x^4}) - 4(x^3 + \frac{1}{x^3}) + 10(x^2 + \frac{1}{x^2}) - 16(x + \frac{1}{x}) + 19 = 0.$$

Now, some computation will yield the following results:

$$\begin{aligned} x^2 + \frac{1}{x^2} &= (x + \frac{1}{x})^2 - 2, \\ x^3 + \frac{1}{x^3} &= (x + \frac{1}{x})^3 - 3(x + \frac{1}{x}), \\ x^4 + \frac{1}{x^4} &= (x + \frac{1}{x})^4 - 4(x^2 + \frac{1}{x^2}) - 6 \\ &= (x + \frac{1}{x})^4 - 4(x + \frac{1}{x})^2 + 2. \end{aligned}$$

Substituting these into our equation as well as $y = x + \frac{1}{x}$ gives

$$\begin{aligned} (y^4 - 4y^2 + 2) - 4(y^3 - 3y) + 10(y^2 - 2) - 16y + 19 &= 0 \\ y^4 - 4y^3 + 6y^2 - 4y + 1 &= 0 \\ (y - 1)^4 &= 0. \end{aligned}$$

Thus, the only solution is $y = 1$. Finally, we have

$$\begin{aligned} x + \frac{1}{x} &= 1 \\ x^2 - x + 1 &= 0 \\ x &= \frac{1 \pm i\sqrt{3}}{2}. \end{aligned}$$

Problem 12B. Proposed by Max Jiang

Instead of an election, Donald and Joe play a grid-coloring game to decide who gets to be the next president. Given a $n \times m$ grid of unit squares, the candidates will take turns coloring $1 \times k$ or $k \times 1$ sub-arrays of the grid, where k is a positive integer, such that no cell in the sub-array is already colored. The last player to move is declared the winner (at which point the grid will be completely colored). Find all pairs (n, m) such that the first player to move has a winning strategy.
Solution:
We claim that the first player has a winning strategy for all (m, n) such that not both numbers are even.

If at least one of m or n is odd, say that is it m without loss of generality, we see that the first player can color the entire middle vertical column, splitting the grid into two disjoint $\frac{m-1}{2} \times n$ grids. Note that at this point, the first player can copy the second player's move on the opposite grid. In this way, the two disjoint grids will always be in the same position after the first player's turn. Thus, the first player must also be the one that leaves both grids fully colored.

Otherwise, both m and n are even. Note that there is a bijection that takes each cell (x, y) to the cell $(m + 1 - x, n + 1 - y)$. This is essentially mapping each cell to its image under a $180°$ rotation of the grid. Since m and n are even, every cell is mapped to a different cell.
We see that after every move that the first player makes, the second player can color the cells colored by the first under the rotation. In this way, the state of each cell and its image must always be the same after the second player's turn. Thus, it is not possible that the image of the first player's move has already been colored in a previous turn.
Additionally, any $1 \times k$ or $k \times 1$ grid does not intersect with its image since they must exist in distinct columns or rows, respectively. Hence, the second player's moves will always be valid.
Since it is always possible for the second player to move after the first player, only he can make the last move meaning he must win.

Problem 13B. Proposed by Alexander Monteith-Pistor

Let $p(x)$ be a polynomial with integer coefficients satisfying

$$p(x + 2020)^{2020} = p(x^{2020}) + 2020$$

Find all possible values of $p(0)$.

Problem 14B. Proposed by Ken Jiang

For integer $n > 1$, determine which expression is greater: $n^{n!}$ or $(n^n)!$.

Solution:

Consider the base-n logarithms of the expressions $\log_n n^{n!} = n!$ and $\log_n (n^n)! = \sum_{i=1}^{n^n} \log_n i$. All but the first n terms in the summation are greater than 1, so it suffices to prove that

$$\begin{aligned} n^n - n &> n! \\ n(n^{n-1} - 1) &> n! \\ n^{n-1} - 1 &> (n-1)! \\ n^{n-1} - 1 > (n-1)^{n-1} &> (n-1)! \end{aligned}$$

Thus, since $\log_n n^{n!} < \log_n (n^n)!$, $n^{n!} < (n^n)!$.

Problem 15B. Proposed by Ken Jiang

Let $f_n(x)$ be a function such that:

$$f_n(x) = \begin{cases} 1 & \text{if } x \leq n, \\ \sum\limits_{i=x-n}^{x-1} f_n(i) & \text{if } x > n. \end{cases}$$

Prove that

$$\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i} = n$$

Solution:

Consider the sum

$$\sum_{j=0}^{n-1} \frac{\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i}}{2^j}$$

This can also be written as

$$\sum_{i=1}^{\infty} \frac{\sum\limits_{j=i-n+1}^{i} f_n(j)}{2^i}$$

Where we let $f_n(j) = 0$ for $j \leq 0$. But we know this is just

$$\sum_{i=n}^{\infty} \frac{f_n(i+1)}{2^i} + \sum_{i=1}^{n-1} \frac{i}{2^i}$$

The first summation in this expression is just double our original sum less a few terms. Specifically,

$$\sum_{i=n}^{\infty} \frac{f_n(i+1)}{2^i} = 2\left(\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i} - \sum_{i=1}^{n-1} \frac{f_n(i)}{2^i}\right)$$

Since $f_n(i) = 1$ for $i < n$, the first few terms sum to $\frac{2^n - 1}{2^n}$. So

$$\sum_{i=n}^{\infty} \frac{f_n(i+1)}{2^i} = 2\left(\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i} - \frac{2^n - 1}{2^n}\right)$$

To evaluate $\sum_{i=1}^{n-1} \frac{i}{2^i}$, we consider its double:

$$2\sum_{i=1}^{n-1} \frac{i}{2^i} = \sum_{i=1}^{n-1} \frac{i}{2^{i-1}} = \sum_{i=0}^{n-2} \frac{i+1}{2^i}$$

$$\sum_{i=1}^{n-1} \frac{i}{2^i} = \sum_{i=0}^{n-2} \frac{i+1}{2^i} - \sum_{i=1}^{n-1} \frac{i}{2^i} = \frac{1}{1} + \sum_{i=1}^{n-2} \frac{1}{2^i} - \frac{n-1}{2^{n-1}}$$

$$\sum_{i=1}^{n-1} \frac{i}{2^i} = 1 + \frac{2^{n-2} - 1}{2^{n-2}} - \frac{n-1}{2^{n-1}} = 2 - \frac{1}{2^{n-2}} - \frac{n-1}{2^{n-1}}$$

Thus,

$$\sum_{j=0}^{n-1} \frac{\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i}}{2^j} = 2\left(\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i} - \frac{2^n - 1}{2^n}\right) + 2 - \frac{1}{2^{n-2}} - \frac{n-1}{2^{n-1}}$$

But we also know that

$$\sum_{j=0}^{n-1} \frac{\sum_{i=1}^{\infty} \frac{f_n(i)}{2^i}}{2^j} = \frac{2^n - 1}{2^{n-1}} \sum_{i=1}^{\infty} \frac{f_n(i)}{2^i}$$

So

$$\begin{aligned}
\frac{2^n-1}{2^{n-1}}\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= 2\left(\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i}-\frac{2^n-1}{2^n}\right)+2-\frac{1}{2^{n-2}}-\frac{n-1}{2^{n-1}}\\
-\frac{1}{2^{n-1}}\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= -\frac{2^n-1}{2^{n-1}}+2-\frac{1}{2^{n-2}}-\frac{n-1}{2^{n-1}}\\
\frac{1}{2^{n-1}}\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= \frac{2^n+1}{2^{n-1}}-2+\frac{n-1}{2^{n-1}}\\
\frac{1}{2^{n-1}}\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= \frac{1}{2^{n-1}}+\frac{n-1}{2^{n-1}}\\
\frac{1}{2^{n-1}}\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= \frac{n}{2^{n-1}}\\
\sum_{i=1}^{\infty}\frac{f_n(i)}{2^i} &= n
\end{aligned}$$

Problem 16B. Proposed by Nicolas Sullivan

Let ΔABC be an equilateral triangle with side length 1. Choose points M, N and P on BC, CA and AB respectively. Find the smallest possible ratio between the area of ΔMNP and ΔABC if

$$64\left[\frac{1}{BM^3}+\frac{1}{CN^3}+\frac{1}{AP^3}\right]+\left[\frac{1}{CM^3}+\frac{1}{AN^3}+\frac{1}{BP^3}\right]=729.$$

Solution:
Firstly, let S_1 be the area of the equilateral triangle, and let S_2 be the area of ΔMNP. Also, let $a = BM$, $b = CN$ and $c = AP$. Since $AB = BC = CA = 1$, then $S_1 = \frac{1}{2}\sin(\pi/3)$. Additionally, we can find S_2 by subtracting the areas of the corner triangles:

$$\begin{aligned}
S_2 &= \frac{1}{2}\sin(\pi/3)(1-a(1-c)-b(1-a)-c(1-b))\\
&= S_1(1-(a+b+c)+(bc+ca+ab))\\
\frac{S_2}{S_1} &= (1-a)(1-b)(1-c)+abc.
\end{aligned}$$

Next, we can express the constraint from the problem statement as

$$64\left[\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right]+\left[\frac{1}{(1-a)^3}+\frac{1}{(1-b)^3}+\frac{1}{(1-c)^3}\right]=729.$$

By the AM-HM inequality, we know that for positive m, n:

$$\begin{aligned}\frac{64m+n}{9} &\geq \frac{9}{8(\frac{1}{8m})+\frac{1}{n}} \\ 64m+n &\geq \frac{81}{\frac{1}{m}+\frac{1}{n}} \\ \frac{1}{m}+\frac{1}{n} &\geq \frac{81}{64m+n}.\end{aligned}$$

Thus, we can say that

$$\frac{1}{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}+\frac{1}{\frac{1}{(1-a)^3}+\frac{1}{(1-b)^3}+\frac{1}{(1-c)^3}} \geq \frac{81}{729}=\frac{1}{9}.$$

Finally, using the HM-GM inequality, we know that

$$\frac{3}{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}} \leq (a^3b^3c^3)^{1/3}=abc,$$

and

$$\frac{3}{\frac{1}{(1-a)^3}+\frac{1}{(1-b)^3}+\frac{1}{(1-c)^3}} \leq ((1-a)^3(1-b)^3(1-c)^3)^{1/3}=(1-a)(1-b)(1-c).$$

Thus, the ratio between areas must satisfy

$$\begin{aligned}\frac{S_2}{S_1} &\geq \frac{3}{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}+\frac{3}{\frac{1}{(1-a)^3}+\frac{1}{(1-b)^3}+\frac{1}{(1-c)^3}} \\ &\geq \frac{3}{9}=\frac{1}{3}.\end{aligned}$$

We can easily verify that $a=b=c=\frac{2}{3}$ satisfies both the original constraint and $\frac{S_2}{S_1}=\frac{1}{3}$, so this the minimum possible ratio is areas is

$$\frac{S_2}{S_1}=\frac{1}{3}.$$

Problem 17B. Proposed by Nikola Milijevic

Recall that the Fibonacci sequence is defined by $f_1=1, f_2=1$ and $f_k=f_{k-1}+f_{k-2}$ for $k \geq 3$. Prove that $f_{n+4} \equiv 2f_n \pmod 3$ for all $n \in \mathbb{N}$

Solution:

We solve this problem with strong induction on n, where $P(n)$ is the statement $f_{n+4} \equiv 2f_n \pmod 3$

Base Case: Consider two base cases, when $n=1$ and $n=2$

We have $f_1=1, f_5=5$ and $5 \equiv 2 \pmod 3$ so $f_5 \equiv 2f_1 \pmod 3$. Similarily, $f_2=1, f_6=8$ and $8 \equiv 2 \pmod 3$ so $f_6 \equiv 2f_2 \pmod 3$. Therefore the base cases hold.

Inductive Step: Let k be an arbitrary integer greater than 2. Assume the inductive hypotheses, $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$, that is, $f_{i+4} \equiv 2f_i \pmod 3$ for $i \in \{1, 2, \ldots, k\}$. We wish to prove the inductive conclusion, $f_{k+5} \equiv 2f_{k+1} \pmod 3$.

$$f_{k+4} \equiv 2f_k (\text{mod } 3) \mid \text{by inductive hypothesis}$$

$$f_{k+3} \equiv 2f_{k-1} (\text{mod } 3) \mid \text{by inductive hypothesis}$$

$$f_{k+4} + f_{k+3} \equiv 2(f_k + f_{k-1}) (\text{mod } 3)$$

$$f_{k+5} \equiv 2f_{k+1} (\text{mod } 3)$$

We have proven the inductive conclusion, so therefore the statement $P(n)$ is true by the principle of strong induction.

Problem 18B. Proposed by Andy Kim

There is an event in Ottawa, which has attendees sitting in a row of n chairs. However, to comply to the social distancing guidelines, there must be a space of at least one empty chair between attendees. Given that there is at least 1 attendee, show that the number of possible seating arrangements is $F_{n+2} - 1$. (note: F_n is the n'th fibonacci number, which is defined by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for natural $n \geq 3$)

Solution:

Let s_n be the number of possible seating arrangements for n chairs. Now, we proceed by induction.

Base case: $n = 1$, $n = 2$

For 1 chair, there is only one possible arrangement, so we have

$$s_1 = 1 = 2 - 1 = F_3 - 1$$

For 2 chairs, a person can sit in either chair, so we have

$$s_2 = 2 = 3 - 1 = F_4 - 1$$

Inductive step:

Suppose we have that $s_k = F_{k+2} - 1$ and $s_{k-1} = F_{k+1} - 1$ for some natural $k \geq 2$. Then, for $k+1$ chairs, if the $k+1$'th chair is not taken, The number of possible arrangements is exactly equal to s_{k-2}.

If the $k+1$'th chair is taken, either there is at least 1 other chair taken, in which case we have s_{k-1} possible arrangements, or it is the only chair taken, in which case we have 1 arrangement.

Then, adding these up, we have

$$s_{k+1} = s_k + s_{k-1} + 1 = F_{k+2} - 1 + F_{k+1} - 1 + 1 = F_{k+3} - 1$$

So, the induction is complete, and we have that $s_n = F_{n+2} - 1$ for all natural n.

Problem 19B. Proposed by Luca Tu

Find the maximum of $\log a \cdot \log c$, given:

$$\log a + \log_b c = 3 \text{ and } \log b + \log_a c = 4$$

(where $\log x$ denotes $\log_{10} x$)
Solution:

By the change of base formula, $\log_b c = \frac{\log c}{\log b} = \frac{z}{y}$. Similarly, $\log_a c = \frac{z}{x}$. Then

$$x + \frac{z}{y} = 3, \quad y + \frac{z}{x} = 4$$
$$xy + z = 3y, \quad xy + z = 4x$$
$$xy + z = 3y = 4x$$

Since $3y = 4x$, we can assume $x = 3t$, $y = 4t$ for some real number t.

$$3t \cdot 4t + z = 3 \cdot 4t$$
$$z = 12t - 12t^2 = 12t(1-t)$$

Multiply z by x, (since $xz = \log a \cdot \log c$):

$$\begin{aligned} xz &= 3t \cdot (12t(1-t)) \\ &= 36t^2 \cdot (1-t) \text{ (Time for tricks)} \\ &= 18 \cdot t^2 \cdot [2 \cdot (1-t)] \\ &= 18 \cdot [t^2 \cdot (2-2t)] \\ &= 18 \cdot [t \cdot t \cdot (2-2t)] \\ &\leq 18 \cdot \left[\frac{t + t + (2-2t)}{3}\right]^3 \text{ (AM-GM inequality)} \\ &\leq 18 \cdot \left(\frac{2}{3}\right)^3 = \frac{16}{3}. \end{aligned}$$

(Note $0 \leq \frac{1}{3} \log a = t$. If $t > 2$ then $xz < 0$. Thus, for our purposes we may assume $0 \leq t \leq 2$ which allows us to use AM-GM as shown above)

Now we have the theoretical maximum, but we need to find actual x, y, z that satisfy this. Since the equality for AM-GM inequality holds when all the elements are equal, we have $t = \frac{2}{3}$. Then $x = 2$, $y = \frac{8}{3}$, $z = \frac{8}{3}$. Which indicates $a = 100$, $b = c = 10^{\frac{8}{3}}$. Checking back, we see that this does indeed work. So the final answer is $\frac{16}{3}$

Problem 20B. Proposed by DC

Given six points A_1, A_2, A_3, A_4, A_5, A_6 on a circle such that $\overset{\frown}{A_1A_2} = \overset{\frown}{A_2A_3}$, $\overset{\frown}{A_3A_4} = \overset{\frown}{A_4A_5}$, and $\overset{\frown}{A_5A_6} = \overset{\frown}{A_6A_1}$. Prove that the three points obtained by intersecting A_4A_6 with A_1A_3, A_2A_4 with A_1A_5, and A_3A_5 with A_2A_6 are collinear.

Solution: Let M_1 be the intersection of A_4A_6 and A_1A_3, M_2 be the intersection of A_3A_5 and A_2A_6, and M_3 be the intersection of A_1A_5 and A_2A_4. Let I be the intersection of A_1A_4, A_2A_5, and A_3A_6, all being angle bisectors in $\Delta A_1A_3A_5$.

By applying Menelaus' Theorem in ΔIA_1A_3 with transversal $A_6A_4M_1$ we obtain:

$$\frac{A_6I}{A_6A_3} \times \frac{M_1A_3}{M_1A_1} \times \frac{A_4A_1}{A_4I} = 1.$$

Again, by applying Menelaus' Theorem in ΔIA_1A_5 with transversal $A_2A_4M_3$ we obtain:

$$\frac{M_3A_1}{M_3A_5} \times \frac{A_2A_5}{A_2I} \times \frac{A_4I}{A_4A_1} = 1.$$

Finally, by applying Menelaus' Theorem in ΔIA_3A_5 with transversal $A_6A_2M_2$ we obtain:

$$\frac{A_6A_3}{A_6I} \times \frac{A_2I}{A_2A_5} \times \frac{M_2A_5}{M_2A_3} = 1.$$

By multiplying all three above relationships, we obtain:

$$\frac{M_1A_3}{M_1A_1} \times \frac{M_3A_1}{M_3A_5} \times \frac{M_2A_5}{M_2A_3} = 1.$$

that is the converse of Menelaus' Theorem for the sides A_1A_3, A_1A_5 and A_3A_5, all sides in $\Delta A_1A_3A_5$ intersected by the transversal $M_1M_2M_3$.