Year 2 - Number 2 - November 2021 Problems

January 31, 2022

Problems

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 67A. Proposed by Eliza Andreea Radu

Consider the convex quadrilateral $ABCD$ and the parallelograms $ACPD$ and $ABDQ$. Find $m(\measuredangle(AC, BD))$ knowing that $BP = 16$, $CQ = 12$, $AC = 4$, and $BD = 7\sqrt{2}$.

Problem 76A. Proposed by Vedaant Srivastava

Find all triples $(x, y, z) \in \mathbb{R}^3$ that satisfy the following system of equations:

$$\begin{cases} x^3 = -3x^2 - 11y + 26 \\ y^3 = 3y - 7z + 23 \\ z^3 = -9z^2 + 13x - 121 \end{cases}$$

Problem 77A. Proposed by Octavian Tiberiu Bacain

Show that $\frac{2x}{2x+4} + \frac{3y}{3y+9} + \frac{5z}{5z+25} \geq 2$ for any x, y, and z integers satisfying $2x + 3y + 5z = 12$.

Solution Problem 77A

The statement is not true; one counterexample is enough.

Problem 78A. Proposed by Aurelia Georgescu

Find x, $y \in \mathbb{Z}$ such that $\frac{4x^2+10x+7}{4y^3-12y^2+8y-1} \in \mathbb{Z}$.

Problem 79A. Proposed by Octavian Tiberiu Bacain

Prove that

$$\frac{2}{3\sqrt{2}} \cdot \frac{2}{5\sqrt{6}} \cdot \frac{2}{7\sqrt{12}} \cdots \frac{2}{201\sqrt{10100}} > \frac{1}{\frac{101!^2}{101}}$$

Solution Problem 79A

We will derive from AM-GM $\frac{a+b}{2} \geq \sqrt{ab}$:

$$\frac{2}{a+b} \leq \frac{1}{\sqrt{ab}} \Rightarrow \frac{2}{(a+b)\sqrt{ab}} \leq \frac{1}{ab}.$$

Observed that all the factors from the initial expression have the following form: $\frac{2}{(n+n+1)\sqrt{n(n+1)}}$. Consequently, their product will be less then

$$\frac{1}{1\times 2}\times\frac{1}{2\times 3}\times\frac{1}{3\times 4}\times\cdots\times\frac{1}{100\times 101} = \frac{1}{1^2\times 2^2\times 3^2\times\cdots\times 100^2\times 100} = \frac{1}{\frac{101!^2}{101}}.$$

Problem 80A. Proposed by Vlad Armeanu

Solve in $\mathbb{R}$ the equation:

$$x + \left[x - \frac{2021}{506}\right]^2 + \left|x^2 - 9x + 20\right| = 2\left(\sqrt{x-3} + 1\right)$$

Solution Problem 80A

From the conditions regarding the existence of the equation:

$$x - 3 \geq 0 \Rightarrow x \in [3, \infty)$$

Observe that:

$$x - 2\left(\sqrt{x-3} + 1\right) + \left[x - \frac{2021}{506}\right]^2 + \left|x^2 - 9x + 20\right| = 0$$

Observe that:

$$\left(\sqrt{x-3} - 1\right)^2 = x - 3 - 2\sqrt{x-3} + 1 = x - 2 - 2\sqrt{x-3} = x - 2\left(1 + \sqrt{x-3}\right)$$

The equation is now:

$$\left(\sqrt{x-3} - 1\right)^2 + \left[x - \frac{2021}{506}\right]^2 + \left|x^2 - 9x + 20\right| = 0$$

$$\left.\begin{array}{l} \left(\sqrt{x-3} - 1\right)^2 \geq 0 \\ \left[x - \frac{2021}{506}\right]^2 \geq 0 \\ \left|x^2 - 9x + 20\right| \geq 0 \\ \text{The sum of three positive number is equal to zero if each number is equal to zero.} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left(\sqrt{x-3}-1\right)^2 = \left[x - \frac{2021}{506}\right]^2 = \left|x^2-9x+20\right| = 0 \Rightarrow$$
$$\Rightarrow \sqrt{x-3}-1=0 \Rightarrow \sqrt{x-3}=1 \Rightarrow x-3=1 \Rightarrow x=4$$

Verify for $x=4$:

$$\left[4-\frac{2021}{506}\right]^2 = 0 \Rightarrow \left[\frac{4*506}{506}-\frac{2021}{506}\right]^2 = 0 \Rightarrow \left[\frac{2024}{506}-\frac{2021}{506}\right]^2 \Rightarrow \left[\frac{3}{506}\right]^2 = 0$$

Problem 81A. Proposed by Octavian Tiberiu Bacain

Find x and y integers such that

$$\left[\frac{y^2(x^2+1)}{x^2+y^2+1} + \frac{x(y^2+1)}{x^2+y^2+1} + \frac{x^4+y^4+2}{x^2+y^2+1}\right] = (x+y)^2 - 2xy + 2$$

Solution Problem 81A

Observe that

$$\left[\frac{y^2(x^2+1)}{x^2+y^2+1} + \frac{x(y^2+1)}{x^2+y^2+1} + \frac{x^4+y^4+2}{x^2+y^2+1}\right] = \left[\frac{x^2+y^2+1)^2}{x^2+y^2+1} + \frac{1}{x^2+y^2+1}\right] =$$
$$= \left[x^2+y^2+1+\frac{1}{x^2+y^2+1}\right] = x^2+y^2+1+\left[\frac{1}{x^2+y^2+1}\right].$$

Consequently,

$$x^2+y^2+1+\left[\frac{1}{x^2+y^2+1}\right] = (x+y)^2-2xy+2$$

and

$$x^2+y^2+1+\left[\frac{1}{x^2+y^2+1}\right] = x^2+2xy+y^2-2xy+2$$
$$x^2+y^2+1+\left[\frac{1}{x^2+y^2+1}\right] = x^2+y^2+1+1$$
$$\left[\frac{1}{x^2+y^2+1}\right] = 1$$

But

$$\left[\frac{1}{x^2+y^2+1}\right] \le 1.$$

Then we have only the equality: $\frac{1}{x^2+y^2+1} = 1$ with $x^2+y^2+1 = 1$ and $x^2+y^2=0$ with solutions $x=0$ and $y=0$.

Problem 82A. Proposed by Matei Neascu

If $a,b,c > 0$ and $a+b+c=4$, find the minimum value of the sum $S = \frac{a+b+3}{c+5} + \frac{b+c+3}{a+5} + \frac{c+a+3}{b+5}$

Solution Problem 82A

Considering the condition $a+b+c=4$, rewrite the numerators of the fractions

$$S=\frac{4-c+3}{c+5}+\frac{4-a+3}{a+5}+\frac{4-b+3}{b+5}=\frac{12-(c+5)}{c+5}+\frac{12-(a+5)}{a+5}+\frac{12-(b+5)}{b+5}$$

$$S=12\left(\frac{1}{c+5}+\frac{1}{a+5}+\frac{1}{b+5}\right)-3 \tag{1}$$

Applying Titus' Lemma:

$$\frac{1}{c+5}+\frac{1}{a+5}+\frac{1}{b+5}\geq\frac{(1+1+1)^2}{\underbrace{a+b+c}_{4}+15}=\frac{9}{19}$$

So, (1) becomes:

$$S\geq 12\cdot\frac{9}{19}-3=\frac{108-57}{19}=\frac{51}{19} \text{ with equality for } a=b=c=\frac{4}{3}$$

Conclusion: $\min(S)=\frac{51}{19}$ is reached for $a=b=c=\frac{4}{3}$.

Problem 83A. Proposed by Gabriel Crisan

Given the points $A(1,4)$, $B(1,0)$, $C(3,2)$, and $M(\frac{\sqrt{10}+2}{2},\frac{\sqrt{6}+4}{2})$, prove that the projections of M on sides AB, BC, and AC are collinear.

Solution Problem 83A

We are trying to find the coordinates of the center of the circumscribed circle of ΔABC. The equations of the sides AB, BC and AC are:

$$AB:\frac{x-1}{1-1}=\frac{y-4}{0-4}\Rightarrow -4x+4=0$$

$$AC:\frac{x-1}{3-1}=\frac{y-4}{2-4}\Rightarrow -x-y+5=0$$

$$BC:\frac{x-1}{3-1}=\frac{y-0}{2-0}\Rightarrow x-y-1=0$$

and the slopes of the sides AC and BC are $m_{AC}=1$ and $m_{BC}=-1$.
Let N and P be the midpoints of sides AC and BC. We have $x_N=\frac{1+3}{2}=2$ and $y_N=\frac{4+2}{2}=3$. Consequently, $N(2,3)$. Similarly, we find $P(2,1)$.
We can calculate the equations of the parallel bisectors of the sides AC and BC:
The parallel bisector of the side AC: $y-3=1(x-2)\Rightarrow -x+y-1=0$
The parallel bisector of the side BC: $y-1=-1(x-2)\Rightarrow x+y-3=0$

We want now to find the center O of the circumscribed circle of ΔABC. Solving the system formed with the last two equations (O is the intersection of the parallel bisectors described above), we found $x = 1$ and $y = 2$. Consequently, $O(1, 2)$.
Let denote r the radius of the circumscribed circle of ΔABC and $r = OA = \sqrt{(1-1)^{+}(4-2)^2} = 2$.
Then we can demonstrate that A, B, C and M are concyclic points:

$$OM = \sqrt{(\frac{\sqrt{10}+2}{2} - 1)^2 + (\frac{\sqrt{6}+4}{2} - 2)^2} = \sqrt{\frac{10}{4} + \frac{6}{4}} = 2 = r = OA.$$

Consequently, the points are concyclic and, using the Simson's Theorem, we demonstrate the conclusion.

Problem 84A. Proposed by Vlad Armeanu

Solve in $\mathbb{R}$ the equation:

$$\sqrt{z - y^2 - 6x - 26} + x^2 + 6y + z - 8 = 0$$

Solution Problem 84A

From the existence condition:

$$z - y^2 - 6x - 26 \geq 0 \qquad (1)$$

From

$$\sqrt{z - y^2 - 6x - 26} \geq 0 \Longrightarrow x^2 + 6y + z - 8 < 0 \qquad (2)$$

we obtain:

$$-z + y^2 + 6x + 26 \leq 0 \qquad (3)$$

By adding the relationships (2) and (3)

$$x^2 + 6y + z - 8 - z + y^2 + 6x + 26 \leq 0$$

$$x^2 + 6x + y^2 + 6y + 18 \leq 0$$

$$x^2 + 6x + 9 + y^2 + 6y + 9 \leq 0$$

$$\left.\begin{array}{l} (x+3)^2 + (y+3)^2 \leq 0 \\ (x+3)^2 = 0 \\ (y+3)^2 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x+3=0 \\ y+3=0 \end{array}\right. \quad \Rightarrow x = y = -3$$

By substituting x and y in the initial equation:

$$\sqrt{z - (-3)^2 - 6*(-3) - 26} + (-3)^2 + 6*(-3) + z - 8 = 0 \Rightarrow$$

$$\sqrt{z - 9 + 18 - 26} + 9 - 18 + z = 0 \Rightarrow \sqrt{z - 17} + z - 17 = 0$$

Hovever

$$z - 17 \geq 0$$

Because

$$z - 17 \leq 0$$

we can conclude $z = 17$.

Problem 85A. Proposed by Ilinca Maria Popa

Solve in $\mathbb{R}$ the equation:

$$\sqrt{x_1 \cdot (2 - x_1)} + \sqrt{x_2 \cdot (4 - x_2)} + \sqrt{x_3 \cdot (8 - x_3)} + \cdots + \sqrt{x_{2021} \cdot (2^{2021} - x_{2021})} = 2^{2021} - 1$$

Solution Problem 85A

By applying AM-GM to the numbers x_1 and $2 - x_1$ we obtain:

$$\sqrt{x_1 \cdot (2 - x_1)} \leq \frac{x_1 + (2 - x_1)}{2} = \frac{2}{2} = 1$$

Similarly,

$$\sqrt{x_2 \cdot (4 - x_2)} \leq \frac{x_2 + (4 - x_2)}{2} = \frac{4}{2} = 2$$

$$\sqrt{x_3 \cdot (8 - x_3)} \leq \frac{x_3 + (8 - x_3)}{2} = \frac{8}{2} = 2^2$$

$$\cdots$$

$$\sqrt{x_{2021} \cdot (2^{2021} - x_{2021})} \leq \frac{x_{2021} + (2^{2021} - x_{2021})}{2} = \frac{2^{2021}}{2} = 2^{2020}$$

By summing the above relationships, we obtain:

$$\sqrt{x_1 \cdot (2 - x_1)} + \sqrt{x_2 \cdot (4 - x_2)} + \sqrt{x_3 \cdot (8 - x_3)} + \cdots + \sqrt{x_{2021} \cdot (2^{2021} - x_{2021})} \leq 1 + 2 + 2^2 + \cdots + 2^{2020} =$$

$$= 2^{2021} - 1.$$

However, from the statement of the problem:

$$\sqrt{x_1 \cdot (2 - x_1)} + \sqrt{x_2 \cdot (4 - x_2)} + \sqrt{x_3 \cdot (8 - x_3)} + \cdots + \sqrt{x_{2021} \cdot (2^{2021} - x_{2021})} = 2^{2021} - 1.$$

Consequently, $x_1 = 2 - x_1$ and $x_1 = 1$, $x_2 = 4 - x_2$ and $x_2 = 2$, $x_3 = 8 - x_3$ and $x_3 = 2^2$, $\cdots$, $x_{2021} = 2^{2021} - x_{2021}$ and $x_{2021} = 2^{2020}$.
To conclude, the only numbers that address the requirements of the problem are $x_1 = 1; x_2 = 2; x_3 = 2^2; \cdots; x_{2021} = 2^{2020}$.

Problem 86A. Proposed by Maria Radu

Find how many values $n \in \mathbb{R} \setminus \mathbb{Q}$ satisfy the condition that both $4n^2 - 6n - 2$ and $4n^3 - 2n^2 - 1$ are simultaneously rational numbers.

Problem 87A. Proposed by Vlad Armeanu

Solve in $\mathbb{R}$ the equation:

$$\left[\frac{3x+11}{15}\right]+\left[\frac{3x+16}{15}\right]=5$$

Solution Problem 87A

Observe that

$$\left[\frac{3x+11}{15}\right]=\left[\frac{x+2}{5}+\frac{1}{3}\right]$$

and

$$\left[\frac{3x+16}{15}\right]=\left[\frac{x+2}{5}+\frac{2}{3}\right]$$

Denote

$$\frac{x+2}{5}=t\Rightarrow\left[t+\frac{1}{3}\right]+\left[t+\frac{2}{3}\right]=5$$

Add [t] and apply Hermite's identity

$$[t]+\left[t+\frac{1}{3}\right]+\left[t+\frac{2}{3}\right]=5+[t]$$

$$[3t]=5+[t]$$

$$t-1<[t]\leq t\Rightarrow$$

$$\left.\begin{array}{l}3t-1<[3t]\leq 3t\\ t+4<[t]+5\leq t+5\end{array}\right\}\Rightarrow$$

$$t+4<3t$$

$$3t-1<t+5$$

Solve the system of inequations:

$$t+4<3t\Rightarrow 4<2t\Rightarrow 2<t \tag{1}$$

$$3t-1<t+5\Rightarrow 2t<6\Rightarrow t<3 \tag{2}$$

From (1) and (2) obtain

$$2<t<3$$

due to

$$t\in(2,3)\Rightarrow[t]=2\Rightarrow[3t]=5+[t]=7\Rightarrow$$

$$7\leq 3t<8\Rightarrow\frac{7}{3}\leq t<\frac{8}{3}$$

By intersecting both intervals :

$$\left.\begin{array}{l} \dfrac{7}{3} \leq t < \dfrac{8}{3} \\ \\ 2 < t < 3 \end{array}\right\} \Rightarrow \frac{7}{3} \leq t < \frac{2}{3}$$

$$\frac{7}{3} \leq \frac{x+2}{5} < \frac{8}{3} \qquad | \cdot 5$$

$$\frac{35}{3} \leq x+2 < \frac{40}{3} \qquad | \cdot 3$$

$$35 \leq 3x+6 < 40 \Rightarrow 29 \leq 3x < 34 \Rightarrow \frac{29}{3} \leq x < \frac{34}{3}$$

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive integers $n_1, ..., n_k$ satisfying

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 56B. Proposed by Alexander Monteith-Pistor

A game is played with white and black pieces and a chessboard (8 by 8). There is an unlimited number of identical black pieces and identical white pieces. To obtain a starting position, any number of black pieces are placed on one half of the board and any number of white pieces are placed on the other half (at most one piece per square). A piece is called matched if its color is the same of the square it is on. If a piece is not matched then it is mismatched. How many starting positions satisfy the following condition

$$\# \text{ of matched pieces} - \# \text{ of mismatched pieces} = 16$$

(your answer should be a binomial coefficient)

Problem 62B. Proposed by Eliza Andreea Radu

If $a_1, a_2, \ldots, a_{2021} \in \mathbb{R}_+$ such that $\sum_{i=1}^{2021} a_i > 2021$, prove that

$$a_1^{2^{2021}} \cdot 1 \cdot 2 + a_2^{2^{2021}} \cdot 2 \cdot 3 + \ldots + a_{2021}^{2^{2021}} \cdot 2021 \cdot 2022 > 4086462.$$

Solution Problem 62B

By using TITU's inequality, we obtain the following relationship:

$$E = a_1^{2^{2021}} \cdot 1 \cdot 2 + a_2^{2^{2021}} \cdot 2 \cdot 3 + \cdots a_{2021}^{2^{2021}} \cdot 2021 \cdot 2022 =$$

$$= \frac{a_1^{2^{2021}}}{\frac{1}{1\cdot 2}} + \frac{a_2^{2^{2021}}}{\frac{1}{2\cdot 3}} + \cdots + \frac{a_{2021}^{2^{2021}}}{\frac{1}{2021\cdot 2022}} \geq \frac{a_1^{2^{2020}} + a_2^{2^{2020}} + \cdots + a_1^{2^{2020}}}{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{2021\cdot 2022}} =$$

$$= \frac{a_1^{2^{2020}} + a_2^{2^{2020}} + \cdots + a_1^{2^{2020}}}{\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2021} - \frac{1}{2022}} = \frac{\left(\sum_{i=1}^{2021} a_1^{2^{2020}}\right)^2}{\frac{1}{1} - \frac{1}{2022}} = \frac{\left(\sum_{i=1}^{2021} a_1^{2^{2020}}\right)^2}{\frac{2021}{2022}}$$

Next, we will prove that

$$\sum_{i=1}^{2021} a_i^{2^k} \geq \frac{\left(\sum_{i=1}^{2021} a_1^{2^{k-1}}\right)^2}{2021}, k \in \mathbb{N}, k \geq 1$$

by using the CBS inequality:

$$(a_1^{2^k} + a_2^{2^k} + \cdots + a_{2021}^{2^k})(1+1+\cdots+1) \geq (a_1^{2^{k-1}} + a_2^{2^{k-1}} + \cdots + a_{2021}^{2^{k-1}})^2 \Rightarrow \sum_{i=1}^{2021} a_i^{2^k} \geq \frac{\left(\sum_{i=1}^{2021} a_1^{2^{k-1}}\right)^2}{2021}$$

Now that we have demonstrated the above inequality, we obtain :

$$\sum_{i=1}^{2020} a_i^{2^k} \geq \frac{\left(\sum_{i=1}^{2021} a_1^{2^{2019}}\right)^2}{2021} \geq \frac{\left(\frac{\left(\sum_{i=1}^{2021} a_i^{2^{2018}}\right)^2}{2021}\right)^2}{2021} \geq \frac{\left(\sum_{i=1}^{2021} a_1^{2^{2018}}\right)^{2^2}}{2021^{1+2}} \geq \cdots \geq$$

$$\geq \frac{\left(\sum_{i=1}^{2021} a_1^{2^1}\right)^{2^{2019}}}{2021^{1+2+\cdots+2^{2018}}} \geq \frac{\left(\frac{\left(\sum_{i=1}^{2021} a_1\right)^2}{2021}\right)^{2^{2019}}}{2021^{1+2+\cdots+2^{2018}}} = \frac{\left(\sum_{i=1}^{2021} a_i\right)^{2^{2020}}}{2021^{1+2+\cdots+2^{2018}}} \geq$$

$$\geq \frac{2021^{2^{2020}}}{2021^{2^{2020}-1}} = 2021 \Rightarrow \left(\sum_{i=1}^{2021} a_i^{2^{2020}}\right) \geq 2021^2$$

$$E \geq \frac{\left(\sum_{i=1}^{2021} a_i^{2^{2020}}\right)^2}{\frac{2021}{2022}} > \frac{2021^2}{\frac{2021}{2022}} = 2021^2 \cdot \frac{2022}{2021} = 2021 \cdot 2022.$$

Therefore, $E > 2021 \cdot 2022 = 4086462$.

Problem 67B. Proposed by Stefan-Ionel Dumitrescu

Consider a cube $ABCDA'B'C'D'$. Point X lies on face $ADD'A'$. Point Y lies on face $ABB'A'$. Point W is a randomly chosen point on the edges, faces, or interior of the cube. If Z is the midpoint of XY, find the probability that W is the midpoint of an AZ.

Problem 70B. Proposed by Daisy Sheng

Triangle ABC is obtuse where $\measuredangle C > 90°$. Show that

$$4r^2 \leq \frac{a^2b^2c^2}{(a+b+c)^2(c^2-a^2-b^2)},$$

where r is the inradius of $\triangle ABC$ and a, b, c represent the length of the sides opposite to $\measuredangle A, \measuredangle B, \measuredangle C$, respectively.

Solution Problem 70B

Multiplying both sides by $(a+b+c)^2 > 0$, we get

$$4r^2(a+b+c)^2 \leq \frac{(abc)^2}{c^2-a^2-b^2}.$$

Rewriting $(a+b+c)^2$ as $4 \cdot \left(\frac{a+b+c}{2}\right)^2 = 4s^2$, where s denotes the semiperimeter, the inequality becomes

$$4r^2 \cdot 4s^2 \leq \frac{(abc)^2}{c^2-a^2-b^2} \Rightarrow 16(rs)^2 \leq \frac{(abc)^2}{c^2-a^2-b^2}.$$

Letting the area of the triangle be S, we know that $S = rs$. So, we have that

$$16S^2 \leq \frac{(abc)^2}{c^2-a^2-b^2} \Rightarrow 1 \leq \left(\frac{abc}{4S}\right)^2 \cdot \frac{1}{c^2-a^2-b^2} \Rightarrow 1 \leq \frac{R^2}{c^2-a^2-b^2}.$$

Law of Cosines yields $c^2 = a^2 + b^2 - 2ab\cos C$. Thus, the inequality becomes

$$1 \leq \frac{R^2}{-2ab\cos C}.$$

By Extended Law of Sines, we have

$$1 \leq \frac{\left(\frac{a}{2\sin A}\right) \cdot \left(\frac{b}{2\sin B}\right)}{-2ab\cos C} \Rightarrow 1 \leq -\frac{1}{8\sin A\sin B\cos C}.$$

Since $\measuredangle C = 180° - \measuredangle A - \measuredangle B$, we know that $\cos C = -\cos(A+B)$. Thus, the inequality becomes

$$1 \leq \frac{1}{8\sin A\sin B\cos(A+B)}.$$

The product to sum identity yields

$$1 \leq \frac{1}{4[\cos(A-B)\cos(A+B) - \cos^2(A+B)]}.$$

Since $\cos(A-B) \leq 1$, where equality occurs when $\measuredangle A = \measuredangle B$, we see based on comparing the denominators that

$$\frac{1}{4[\cos(A-B)\cos(A+B) - \cos^2(A+B)]} \geq \frac{1}{4[\cos(A+B) - \cos^2(A+B)]}.$$

We know that

$$\frac{1}{4[\cos(A+B) - \cos^2(A+B)]} = \frac{1}{4\cos(A+B)(1-\cos(A+B))}.$$

Since $\measuredangle A + \measuredangle B < 90^\circ \Rightarrow \cos(A+B), 1-\cos(A+B) > 0$. Via AM-GM, we get that

$$\frac{\cos(A+B) + (1-\cos(A+B))}{2} \geq \sqrt{\cos(A+B)\cdot(1-\cos(A+B))}$$

$$\Downarrow$$

$$\frac{1}{4} \geq \cos(A+B)(1-\cos(A+B))$$

$$\Downarrow$$

$$\frac{1}{4\cos(A+B)(1-\cos(A+B))} \geq 1,$$

where equality occurs when $2\cos(A+B) = 1 \Rightarrow A+B = 60^\circ$. We thus know that

$$\frac{1}{4[\cos(A-B)\cos(A+B) - \cos^2(A+B)]} \geq 1,$$

meaning that the original inequality is true. We find the equality case by solving the system

$$\begin{aligned} A &= B \\ A+B &= 60^\circ \\ A+B+C &= 180^\circ, \end{aligned}$$

which yields $A = 30^\circ, B = 30^\circ$, and $C = 120^\circ$.

Problem 76B. Proposed by Alexander Monteith-Pistor

Let $ABCD$ be a quadrilateral with $\measuredangle ABC = 90^\circ$. Points E and F are on AD and BC respectively such that AB is parallel to EF. Further, AC, BD and EF intersect at O. Given that $BF = 4$, $AB = 9$, $AE = 5$ and $CD = 20$, find a polynomial $p(x)$ such that one of its roots is at $x = \frac{DO}{OB}$.

Problem 77B. Proposed by Andy Kim

(i) Evaluate

$$\binom{n}{0} - 2\binom{n}{1} + \cdots \pm 2^n\binom{n}{n} = \sum_{i=0}^{n}(-1)^i 2^i \binom{n}{i}$$

for $n \in \mathbb{Z}_+$.
(ii) Prove that

$$\sum_{i=0}^{n}(-1)^{n-i} i^n \binom{n}{i} = n!$$

for all $n \in \mathbb{Z}_+$.

Problem 78B. Proposed by Ciurea Pavel

Given the positive real numbers x, y, and z, prove that

$$2(\sum_{cyc} x)\sqrt{\sum_{cyc}\sqrt{x^2+y^2+z^2}} \geq$$

$$\geq \sum_{cyc}\sqrt{3(x+y)(x+z)(\sqrt{x^2+y^2+xy}+\sqrt{x^2+z^2+xz}-\sqrt{y^2+z^2+yz}}).$$

Problem 79B. Proposed by Alexandru Benescu

Prove that $3S$ has at least 16 natural divisors, where

$$S = \left[\sqrt{1\cdot 2\cdot 3\cdot 4}\right] + \left[\sqrt{2\cdot 3\cdot 4\cdot 5}\right] + \cdots + \left[\sqrt{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}\right]$$

and $n > 15$, with $n \in \mathbb{N}$.

Solution Problem 79B

We will prove that

$$\left[\sqrt{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}\right] = n\cdot(n+3)$$

$$n\cdot(n+1)\cdot(n+2)\cdot(n+3) = (n^2+3n)(n^2+3n+2) = (n^2+3n+1)^2 - 1$$

But, $(n^2+3n)^2 < (n^2+3n+1)^2 - 1 < (n^2+3n+1)^2$, for each $n > 0$. Thus, $\left[\sqrt{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}\right] = n^2+3n$.
Consequently, $S = 1\cdot 4 + 2\cdot 5 + \cdots + n\cdot(n+3)$ and

$$S = 1^2+2^2+\cdots+n^2+3\cdot(1+2+\cdots+n) = \frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2} = \frac{n(n+1)(2n+10)}{6}$$

$$S = \frac{n(n+1)(n+5)}{3}$$

and $3S = n(n+1)(n+5)$.

Obviously, n and $n+1$ cannot be prime simultaneously ($n > 15$). Thus, at least one of n and $n+1$ is
(i) equal with $d_1 \cdot d_2$ where d_1 is prime and d_2 is natural, greater than 1 and not equal with d_1 or is
(ii) equal with d^2 where d is prime, greater than 2.

In the first case, let $X = d_1 \cdot d_2$. Consequently, X has at least 4 divisors $(1, d_1, d_2, d_1 \cdot d_2)$ and $n(n+1)$ has at least 8 divisors (we used the fact that $\gcd(n, n+1) = 1$ and the Lemma:
If $N = a_1^{b_1} \cdot a_2^{b_2} \cdots a_p^{b_p}$ with $a_1, a_2, \cdots, a_p$ are distinct prime numbers then N has $(b_1+1)(b_2+1)\cdots(b_p+1)$ natural divisors.

Now, we will prove that $\gcd(n(n+1), n+5) = 1$, when $n > 5$.
Let z be the $\gcd(n(n+1), n+5)$ then $z|n^2+n$ and $z|n+5$. Also $z|n^2+5n$. So, $z|4n$. But, from $z|n+5 \Rightarrow z|4n+20 \Rightarrow z|20$.

We want to prove that z cannot be equal with $n+5$, which would mean that there is a prime factor which divides $n+5$ and which does not divide $n(n+1)$. But, $z \leq 20$ and $n+5 > 20 \Rightarrow z \neq n+5$. Thus, using again the lemma mentioned before, we can conclude that $n(n+1)(n+5)$ has at least $8 \cdot 2 = 16$ divisors.
In the second case, let $X = d^2$, d prime and greater than 2 then d^2 is odd and X is odd.

If $X = n$, let Y be $n+1$. If $X = n+1$, let Y be n. Thus, Y is even and obviously greater than 2. Consequently, similarly to the first case, $n(n+1)$ has at least 8 divisors.
Now, we can prove in the same way as for the first case that $n(n+1)(n+5)$ has at least $8 \cdot 2 = 16$ divisors.
Finally, following these demonstrations, we can conclude that $3S$ has at least 16 natural divisors.

Problem 80B. Proposed by Pavel Ciurea

Given the positive real numbers x, y, z and t, prove that:

$$\frac{2xy}{yt}+\frac{2yt}{xz}+\frac{y}{z}+\frac{z}{y}+2\sqrt{(\frac{x}{y}+\frac{y}{x})(\frac{z}{t}+\frac{t}{z})(\frac{y}{z}+\frac{z}{y}-1)(\frac{x}{t}+\frac{t}{x}+1)} \geq \frac{x}{t}+\frac{t}{x}+\sqrt{3}(\frac{x}{z}+\frac{z}{x}+\frac{y}{t}+\frac{t}{y})+4.$$

Problem 81B. Proposed by Alexandru Benescu

Let ABC be a triangle, H its orthocenter and X, Y, Z the circumscribed circles of ΔBHC, ΔAHC, and ΔAHB respectively. Let DE be the common tangent to X and Y, EF to Y and Z, and FD to X and Z, such that all three circles X, Y, Z are inside ΔDEF. Prove that AD, BE and CF are concurrent.

Problem 82B. Proposed by Alexandru Benescu

Prove that:

$$(a+b+c+2)^2+\frac{5}{2}(a+b)(b+c)(c+a)+2(a^3+2)(b^3+2)(c^3+2)+1\geq 100abc$$

where a, b, $c\in\mathbb{R}_+$ and $a^2+b^2+c^2=3$.

Problem 83B. Proposed by Vlad Armeanu

Solve in $\mathbb{R}$ the following equation:

$$\frac{10}{x-10}+\frac{11}{x-11}++\frac{12}{x-12}+\frac{13}{x-13}=2x^2-23x-4.$$

Solution Problem 83B

By adding the terms:

$$\frac{10}{x-10}+1+\frac{11}{x-11}+1+\frac{12}{x-12}+1+\frac{13}{x-13}+1=2x^2-23x$$

$$\frac{10+x-10}{x-10}+\frac{11+x-11}{x-11}+1+\frac{12+x-12}{x-12}+1+\frac{13+x-13}{x-13}+1=2x^2-23x$$

$$\frac{x}{x-10}+\frac{x}{x-11}+\frac{x}{x-12}+\frac{x}{x-13}=2x^2-23x$$

$$x\left(\frac{1}{x-10}+\frac{1}{x-11}+\frac{1}{x-12}+\frac{1}{x-13}\right)=x(2x-23)$$

One solution is $x=0$.

$$\frac{1}{x-10}+\frac{1}{x-11}+\frac{1}{x-12}+\frac{1}{x-13}=2x-23$$

By grouping the terms:

$$\left(\frac{1}{x-10}+\frac{1}{x-13}\right)+\left(\frac{1}{x-11}+\frac{1}{x-12}\right)=2x-23$$

$$\frac{x-13}{(x-10)(x-13)}+\frac{x-10}{(x-13)(x-10)}+\frac{x-12}{(x-11)(x-12)}+\frac{x-11}{(x-11)(x-11)}=2x-23$$

$$\frac{x-13+x-10}{(x-10)(x-13)}+\frac{x-12+x-11}{(x-11)(x-12)}=2x-23$$

$$\frac{2x-23}{(x-10)(x-13)}+\frac{2x-23}{(x-11)(x-12)}=2x-23$$

$$(2x-23)\left(\frac{1}{(x-10)(x-13)}+\frac{1}{(x-11)(x-12)}\right)=2x-23 \qquad \mid :(2x-23)$$

One solution is

$$x = \frac{23}{2}.$$

$$\frac{1}{(x-10)(x-13)} + \frac{1}{(x-11)(x-12)} = 1$$

$$\frac{1}{x^2-13x-10x+130} + \frac{1}{x^2-23x+121} = 1$$

Denote

$$x^2-23x+121 = t$$

$$\frac{1}{t+9} + \frac{1}{t} = 1$$

and

$$\frac{t+t+9}{t(t+9)} = 1 \Rightarrow \frac{2t+9}{t(t+9)} = 1 \Rightarrow$$

$$2t+9 = t^2+9t \Longrightarrow t^2+7t-7 = 0$$

Solve the second order equation is $a = 1$, $b = 7$, $c = -7$

$$\Delta = b^2-4ac \Rightarrow \Delta = 7^2-4(-7)$$

$$\Rightarrow \Delta = 49+28 \Rightarrow \Delta = 77$$

$$x_{1,2} = \frac{-7 \pm \sqrt{77}}{2} \Rightarrow$$

$$x_1 = \frac{\sqrt{77}-7}{2}$$

şi

$$x_2 = \frac{\sqrt{77}+7}{2}$$

Problem 84B. Proposed by Nicholas Sullivan

Let C be a circle of radius r centred at the origin. Consider $n \geq 2$ points on C, $\{P_k : 1 \leq k \leq n\}$, such that their centroid is at the origin. Show that for any point Q on C, the average of the squared lengths $\{\overline{QP_k}\}$ is equal to $2r^2$. That is:

$$\frac{1}{n}\sum_{k=1}^{n}(\overline{QP_k})^2 = 2r^2.$$

Solution Problem 84B

We start by expressing this problem using the complex plane. If each point is expressed as a complex number, then we would like to show that

$$\frac{1}{n}\sum_{k=1}^{n}|q-p_k|^2 = 2r^2,$$

if $|q| = r$ and $|p_k| = r$, with the center of mass condition requiring:

$$\sum_{k=1}^{n} p_k = 0.$$

Expanding the first expression, we have:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}|q-p_k|^2 &= \frac{1}{n}\sum_{k=1}^{n}[|q|^2 + |p_k|^2 - qp_k^* - q^*p_k] \\ &= \frac{1}{n}\sum_{k=1}^{n}[r^2 + r^2 - qp_k^* - q^*p_k].\end{aligned}$$

Next, separating the summation, we have:

$$\frac{1}{n}\sum_{k=1}^{n}|q-p_k|^2 = 2r^2 - \frac{q}{n}\sum_{k=1}^{n}p_k^* - \frac{q^*}{n}\sum_{k=1}^{n}p_k.$$

Since $\sum_{k=1}^{n} p_k = 0$ and thus $\sum_{k=1}^{n} p_k^* = 0$, then:

$$\frac{1}{n}\sum_{k=1}^{n}|q-p_k|^2 = 2r^2 - 0 = 2r^2.$$

This completes the proof.

Problem 85B. Proposed by Daisy Sheng

We have two sequences of numbers. The first sequence is defined by $a_1 = 4$, $a_2 = 12$, and $a_{n+2} = 2a_{n+1} - a_n + 4$, where $n \in \mathbb{Z}^+$. The second sequence is defined by $b_n = 4n^3 + d_1 \cdot n^2 + d_2 \cdot n + d_3$, where $n \in \mathbb{Z}^+$ and d_1, d_2, d_3 represent the coefficients of the polynomial. The polynomial for b_n has roots r_1, r_2, r_3 that satisfy $r_1r_2 + r_1r_3 = \frac{1}{2}$, $r_1r_2r_3 = -\frac{1}{4}$, and $r_2 - r_3 = i$. Prove that a_n and b_n are relatively prime for all $n \in \mathbb{Z}^+$.

Solution Problem 85B

We first find a formula for a_n that is only dependent on the value of n. We are given that

$$a_{n+2} = 2a_{n+1} - a_n + 4 \Rightarrow (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = 4.$$

Since the above indicates that the second finite difference is constant, we see that a_n is defined by a quadratic. We use the fact that

$$\textit{constant finite difference} = \textit{leading coefficient} \cdot \textit{(degree of polynomial)!}$$

to help us find the leading coefficient of the quadratic for a_n. Since $\frac{4}{2!} = 2$, we get that

$$a_n = 2n^2 + c_1 n + c_2,$$

where c_1 and c_2 represent the coefficient and constant term, respectively, for the quadratic. Plugging in $a_1 = 4$ and $a_2 = 12$, we have that

$$\begin{aligned} a_1 &= 2 + c_1 + c_2 = 4, \\ a_2 &= 8 + 2c_1 + c_2 = 12. \end{aligned}$$

Solving this system yields $c_1 = 2$ and $c_2 = 0$. Thus, $a_n = 2n^2 + 2n$.

We now look for the values of d_1, d_2, d_3 in $b_n = 4n^3 + d_1 \cdot n^2 + d_2 \cdot n + d_3$. We are given that

$$r_1 r_2 + r_1 r_3 = \frac{1}{2}, \tag{1}$$

$$r_2 - r_3 = i, \tag{2}$$

$$r_1 r_2 r_3 = -\frac{1}{4}. \tag{3}$$

Rearranging (1) gives us $r_2 + r_3 = \frac{1}{2r_1}$. Combining this result with (2) using elimination gives us

$$\begin{aligned} r_2 &= \frac{1}{4r_1} + \frac{i}{2}, \\ r_3 &= \frac{1}{4r_1} - \frac{i}{2}. \end{aligned}$$

Substituting this into (3), we get

$$r_1 \left(\frac{1}{4r_1} + \frac{i}{2} \right) \left(\frac{1}{4r_1} - \frac{i}{2} \right) = -\frac{1}{4}.$$

Applying difference of squares, we get

$$r_1 \left(\frac{1}{16r_1^2} + \frac{1}{4} \right) = -\frac{1}{4}.$$

Multiplying both sides by $16r_1$ and rearranging gives us

$$4r_1^2 + 4r_1 + 1 = 0 \Rightarrow (2r_1 + 1)^2 = 0 \Rightarrow r_1 = -\frac{1}{2}.$$

Thus, $r_2 = -\frac{1}{2} + \frac{i}{2}$ and $r_3 = -\frac{1}{2} - \frac{i}{2}$.

We now use Vieta's and elementary symmetric polynomials to find d_1, d_2, d_3. We get that:

$$\begin{aligned}\frac{d_1}{4} &= -(r_1 + r_2 + r_3)\\ &= -\left(-\frac{1}{2} + \left(-\frac{1}{2} + \frac{i}{2}\right) + \left(-\frac{1}{2} - \frac{i}{2}\right)\right)\\ &= \frac{3}{2} \Rightarrow d_1 = 6\end{aligned}$$

$$\begin{aligned}\frac{d_2}{4} &= (r_1 r_2 + r_1 r_3) + r_2 r_3\\ &= \frac{1}{2} + \left(-\frac{1}{2} + \frac{i}{2}\right) \cdot \left(-\frac{1}{2} - \frac{i}{2}\right)\\ &= 1 \Rightarrow d_2 = 4\end{aligned}$$

$$\begin{aligned}\frac{d_3}{4} &= -(r_1 r_2 r_3)\\ &= \frac{1}{4} \Rightarrow d_3 = 1.\end{aligned}$$

Therefore, $b_n = 4n^3 + 6n^2 + 4n + 1$.

In order to show that a_n and b_n are relatively prime, we need to prove that $\gcd(a_n, b_n) = \gcd(2n^2 + 2n, 4n^3 + 6n^2 + 4n + 1) = 1$ for all $n \in \mathbb{Z}^+$.

We see that $2n^2 + 2n$ is a multiple of 2 whereas $4n^3 + 6n^2 + 4n + 1 = 2(2n^3 + 3n^2 + 2n) + 1$ is not. Thus,

$$\gcd(2n^2 + 2n, 4n^3 + 6n^2 + 4n + 1) = \gcd(n^2 + n, 4n^3 + 6n^2 + 4n + 1).$$

Applying Extended Euclidean Algorithm gives us:

$$\begin{aligned}\gcd(n^2 + n, 4n^3 + 6n^2 + 4n + 1) &= \gcd(n^2 + n, 4n^3 + 6n^2 + 4n + 1 - 4n(n^2 + n))\\ &= \gcd(n^2 + n, 2n^2 + 4n + 1)\\ &= \gcd(n^2 + n, 2n^2 + 4n + 1 - 2(n^2 + n))\\ &= \gcd(n^2 + n, 2n + 1).\end{aligned}$$

Since $2n + 1$ is not a multiple of n, we see that $\gcd(n^2 + n, 2n + 1) = \gcd(n + 1, 2n + 1)$. Applying Euclidean Algorithm, we get that

$$\gcd(n + 1, 2n + 1) = \gcd(n + 1, n) = \gcd(1, n) = 1.$$

Since $\gcd(a_n, b_n) = 1$ for any $n \in \mathbb{Z}^+$, we have shown that they are relatively prime.

Problem 86B. Proposed by Nicholas Sullivan

Suppose there are 2021 people sitting at a very large circular table. How many ways are there to give each person a red, blue or green hat such that no two neighbouring people have the same colour hat?

Problem 87B. Proposed by Max Jiang

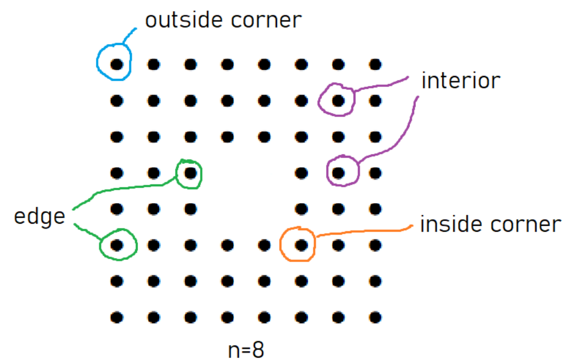
There is a row of n pies, numbered 1 to n from left to right. You start at pie 1 and go rightward. At each pie, there is a $1/2$ chance that you eat the pie, after which you move on to the next uneaten pie (which depends on the direction you are moving). Upon reaching the last uneaten pie, you change the direction you are going (if the last pie still uneaten you will "repeat" it). You continue until all pies are eaten. What is the probability that pie 1 is the last pie you eat? Express your answer as a finite sum in terms of n.

Problem 88B. Proposed by Daisy Sheng

Pierre is coloring a $n \times n$ square grid, where n is even and $n \geq 8$. He chooses to omit the centre 2×2 square grid (see diagram for an example). In the spirit of the holidays, Pierre is coloring the dots either red or green. He also connects horizontally, vertically, and diagonally adjacent dots with lines using the following color scheme rules:

- 2 red dots are connected by a gold line segment.
- 2 green dots are connected by a blue line segment.
- A red and green dot are connected by a silver line segment.

Let there be: r_0 red outside corner dots, r_1 red inside corner dots, r_2 red edge dots, and r_3 red interior dots. If B is the overall number of blue line segments and G is the overall number of gold line segments, find G in terms of B, n, r_0, r_1, r_2, and r_3.



Problem 89B. Proposed by Arnab Sanyal

Let ω_1 and ω_2 be two intersecting circles. Let $\omega_1 \cap \omega_2 = \{P, Q\}$. Let the tangent to ω_1 and ω_2 at P meet ω_1 and ω_2 respectively at B and A. Let the circumcircle of ΔPBA be ω_0. AQ meets ω_0 (possibly extended) at Y and BQ meets ω_0 (possibly extended) at X. Assuming $P \neq Q$, prove or disprove the following statements:
(i) $XABY$ is an isosceles trapezoid;
(ii) $QOBA$ is cyclic, where O is the circumcenter of ω_0.