Year 2 - Number 3 - February 2022 Problems

May 25, 2022

Problems

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 67A. Proposed by Eliza Andreea Radu

Consider the convex quadrilateral $ABCD$ and the parallelograms $ACPD$ and $ABDQ$. Find $m(\measuredangle(AC, BD))$ knowing that $BP = 16$, $CQ = 12$, $AC = 4$, and $BD = 7\sqrt{2}$.

Problem 76A. Proposed by Vedaant Srivastava

Find all triples $(x, y, z) \in \mathbb{R}^3$ that satisfy the following system of equations:

$$\begin{cases} x^3 = -3x^2 - 11y + 26 \\ y^3 = 3y - 7z + 23 \\ z^3 = -9z^2 + 13x - 121 \end{cases}$$

Problem 78A. Proposed by Aurelia Georgescu

Find x, $y \in \mathbb{Z}$ such that $\frac{4x^2+10x+7}{4y^3-12y^2+8y-1} \in \mathbb{Z}$.

Problem 86A. Proposed by Radu Maria

Find how many values $n \in \mathbb{R} \setminus \mathbb{Q}$ satisfy the condition that both $4n^2 - 6n - 2$ and $4n^3 - 2n^2 - 1$ are simultaneously rational numbers.

Solution Problem 86A

We let:

$$4n^2 - 6n - 2 = p, \tag{1}$$
$$4n^3 - 2n^2 - 1 = q, \tag{2}$$

where $p, q \in \mathbb{Q}$.

Multiplying the first equation by n, we get $4n^3 - 6n^2 - 2n = pn$. Subtracting the second equation from this, we get

$$\begin{aligned} pn - q &= (4n^3 - 6n^2 - 2n) - (4n^3 - 2n^2 - 1) \\ &= -4n^2 - 2n + 1. \end{aligned} \tag{3}$$

Adding equations (1) and (3), we get

$$-8n - 1 = p + pn - q$$
$$\Downarrow$$
$$-8n - pn = p - q + 1$$
$$\Downarrow$$
$$n(-8 - p) = p - q + 1.$$

Given that $p, q \in \mathbb{Q}$, we know that $p - q + 1 \in \mathbb{Q}$. From the above equation, we deduce that $n(-8-p) \in \mathbb{Q}$. In addition, since $p \in \mathbb{Q}$, we know that $(-8-p) \in \mathbb{Q}$.

If $(-8-p) \neq 0$, we see that $n(-8-p) \in \mathbb{Q}$ cannot be satisfied given the problem's condition that $n \in \mathbb{R} \setminus \mathbb{Q}$.

Therefore, $(-8 - p) = 0 \Rightarrow p = -8$.

Substituting this into equation (1), we get that

$$4n^2 - 6n - 2 = -8 \Rightarrow 4n^2 - 6n + 6 = 0.$$

However, the determinant is $6^2 - 4 \cdot 4 \cdot 6 < 0$, meaning that there are no real roots and therefore no irrational ones. Thus, n has 0 possible values.

Problem 88A. Proposed by Nicholas Edmond Teler

Solve in $\mathbb{Z}$ the equation:

$$(36x^2 + 204x + 289)(3x + 11)(x + 2) = -2028$$

Solution Problem 88A

Factoring the first parenthesis and multiplying both sides of the equation by 12, we have

$$(6x+17)^2(6x+22)(6x+12) = -24336.$$

Setting $a = 6x + 17$, the equation becomes

$$a^2(a+5)(a-5) = -24336 \Leftrightarrow a^2(a^2-25) = -24336.$$

Letting $m = a^2$, we get that

$$m(m-25) = -24336 \Leftrightarrow m^2 - 25m + 24336 = 0.$$

Applying the quadratic formula, we obtain:

$$(m-169)(m-144) = 0.$$

Therefore, $m = 169$ or $m = 144$.

When $m = 169$, we get that $a = \pm 13$, giving us that $x = -\frac{2}{3}$ or $x = -5$.

When $m = 144$, we get that $a = \pm 12$, giving us that $x = -\frac{5}{6}$ or $x = -\frac{29}{6}$.

Because $x \in \mathbb{Z}$, we conclude that $x = -5$.

Problem 89A. Proposed by Vlad Armeanu

Solve in $\mathbb{R}^+$ the system equations:

$$\begin{cases} a \cdot b \cdot c = 125 \\ (a+b+c)^3\left(\frac{1}{a^3+1000} + \frac{1}{b^3+1000} + \frac{1}{c^3+1000}\right) \leq 9 \end{cases}$$

Solution Problem 89A

We will start with the following identity:

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(a+c)$$

From AM-GM:

$$a + b \geq 2\sqrt{a \cdot b}$$
$$a + c \geq 2\sqrt{a \cdot c}$$
$$b + c \geq 2\sqrt{b \cdot c}$$

By multiplying the above relationships:

$$(a+b)(a+c)(b+c) \geq 8\sqrt{a^2 \cdot b^2 \cdot c^2} \Rightarrow$$

$$\left.\begin{array}{l} (a+b)(a+c)(b+c) \geq 8 \cdot a \cdot b \cdot c \\ a \cdot b \cdot c = 125 \end{array}\right\} \Rightarrow$$

$$(a+b)(a+c)(b+c) \geq 8 \cdot 125 \Rightarrow (a+b+c)^3 \geq a^3+b^3+c^3+3 \cdot 8 \cdot 125 \Rightarrow$$
$$(a+b+c)^3 \geq a^3+b^3+c^3+3000$$

We will prove that:

$$S = (a+b+c)^3 \left(\frac{1}{a^3+1000} + \frac{1}{b^3+1000} + \frac{1}{c^3+1000} \right) \geq 9$$

We have:

$$S \geq (a^3+b^3+c^3+3000) \left(\frac{1}{a^3+1000} + \frac{1}{b^3+1000} + \frac{1}{c^3+1000} \right)$$

We will use the following notations:

$$a^3+1000 = x$$
$$b^3+1000 = y$$
$$c^3+1000 = z$$

From AM-GM:

$$\frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \Leftrightarrow (x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9$$

$$(a^3+1000+b^3+1000+c^3+1000) \left(\frac{1}{a^3+1000} + \frac{1}{b^3+1000} + \frac{1}{c^3+1000} \right) \geq 9$$
$$\Rightarrow S \geq 9$$

and

$$S \leq 9 \Rightarrow S = 9 \Rightarrow a^3+1000 = b^3+1000 = c^3+1000 \Rightarrow a=b=c$$

but

$$a \cdot b \cdot c = 125 \Rightarrow a^3 = 125 \Rightarrow a=b=c=5$$

This is the an unique solution in $\mathbb{R}^+$.

Problem 90A. Proposed by Matei Gheamalinga

Solve for x and y in $\mathbb{Z}$ the system equations:

$$\begin{cases} 9x^2+x^2y^2-6y=0 \\ x^5+4y^2-16y+17=0 \end{cases}$$

Solution Problem 90A

Doing partial factorization on the first equation, we get that

$$9x^2 + x^2y^2 - 6y = 0$$
$$\Downarrow$$
$$x^2(9 + y^2) = 6y.$$

Thus, $x^2 = \frac{6y}{9+y^2}$.

Since $(y-3)^2 \geq 0$ by the Trivial Inequality, we know that

$$y^2 - 6y + 9 \geq 0$$
$$\Downarrow$$
$$\frac{6y}{y^2+9} \leq 1$$
$$\Downarrow$$
$$x^2 \leq 1.$$

Therefore, we have that

$$-1 \leq x \leq 1. \tag{1}$$

From the 2nd equation, we have that

$$\begin{aligned} 0 &= x^5 + 4y^2 - 16y + 17 \\ &= x^5 + 4y^2 - 16y + 1 + 16 \\ &= x^5 + 1 + 4y^2 - 16y + 16 \\ &= x^5 + 1 + 4(y^2 - 4y + 4) \\ &= x^5 + 1 + 4(y-2)^2. \end{aligned}$$

Since $4(y-2)^2 \geq 0$ via the Trivial Inequality, we conclude that $x^5 + 1 \leq 0$. Thus,

$$x^5 \leq -1 \Rightarrow x \leq -1. \tag{2}$$

Combining (1) and (2), we see that $x = -1$. Plugging this in, we get that

$$(-1)^5 + 1 + 4(y-2)^2 = 0$$
$$\Downarrow$$
$$4(y-2)^2 = 0$$
$$\Downarrow$$
$$y - 2 = 0.$$

Therefore, $(x, y) = (-1, 2)$.

Problem 91A. Proposed by Vlad Armeanu

Solve in $\mathbb{R}$ the system equations:

$$\begin{cases} 5^x + 3^x = 6y + 2 \\ 5^y + 3^y = 6z + 2 \\ 5^z + 3^z = 6x + 2 \end{cases}$$

Solution Problem 91A

Without loss of generality, we can assume

$$x \le y \le z$$

$$\Longrightarrow 5^x + 3^x \le 5^y + 3^y \Rightarrow$$

$$\Longrightarrow 6y + 2 \le 6z + 2 \Longrightarrow$$

$$\Longrightarrow y \le z \Longrightarrow$$

$$\Longrightarrow 5^y + 3^y \le 5^z + 3^z \Rightarrow$$

$$\Longrightarrow 6z + 2 \le 6x + 2 \Longrightarrow$$

$$\Longrightarrow z \le x$$

The result contradicts the initial assumption.
Similarly, for a different order we will obtain a contradiction. Consequently $x = y = z$. Thus:

$$5^x + 3^x = 6x + 2$$

$$f(x) = 5^x + 3^x, f : R \to R$$

$$g(x) = 6x + 2, g : R \to R$$

$$\begin{cases} \text{f is a sum of two exponent functions} \\ \text{g is a linear function} \end{cases}$$

Consequently the equation $f(x) = g(x)$ has a maximum of two intersection points.
We can observe that $x = 0$ verifies the equation:

$$5^0 + 3^0 = 6 * 0 + 2 \Longrightarrow 2 = 2$$

Similarly, we can observe that $x = 1$ verifies the equation:

$$5^1 + 3^1 = 6 * 1 + 2 \Longrightarrow 8 = 8$$

The solutions are:

$$I : x = 0; y = 0; z = 0$$

$$II : x = 1; y = 1; z = 1$$

Problem 92A. Proposed by Denisa Dragomir

Solve in $\mathbb{R}$ the equation:

$$\left[\frac{x^2+2x+3}{x^2+2}\right] = \sqrt{x-2\sqrt{x-1}}$$

Solution Problem 92A

We observe that:

$$(\sqrt{x-1}-1)^2 = x-1+1-2\sqrt{x-1} = x-2\sqrt{x-1}.$$

Thus, $x-2\sqrt{x-1}$ is a perfect square.

Our equation thus becomes

$$\left[\frac{x^2+2x+3}{x^2+2}\right] = \left|\sqrt{x-1}-1\right|.$$

In addition, we see that $x-1 \geq 0 \Rightarrow x \geq 1$ in order for the square root to be real.

We will now prove that $\frac{1}{2} \leq \frac{x^2+2x+3}{x^2+2} \leq 2$.

Note from AEs: The proof of the inequality has been edited to become a forwards solution.

We begin by proving that $\frac{x^2+2x+3}{x^2+2} \geq \frac{1}{2}$. From Trivial Inequality, we know that

$$(x+2)^2 \geq 0$$

$$\Downarrow$$

$$x^2+4x+4 \geq 0$$

Adding x^2+2 to both sides, the inequality becomes

$$2x^2+4x+6 \geq x^2+2$$

$$\Downarrow$$

$$2(x^2+2x+3) \geq x^2+2.$$

Since $2(x^2+2) > 0$, we divide it from both sides to get $\frac{x^2+2x+3}{x^2+2} \geq \frac{1}{2}$.

We will now prove that $\frac{x^2+2x+3}{x^2+2} \leq 2$.

From Trivial Inequality, we know that

$$(x-1)^2 \geq 0$$

$$\Downarrow$$

$$x^2 - 2x + 1 \geq 0$$

Adding $x^2 + 2x + 3$ to both sides, we have

$$2x^2 + 4 \geq x^2 + 2x + 3.$$

Since $x^2 + 2 > 0$, we can divide it from both sides to give $2 \geq \frac{x^2+2x+3}{x^2+2}$.

We let $E(x) = \frac{x^2+2x+3}{x^2+2}$. Since $E(x) \in \left[\frac{1}{2}, 2\right]$, we know that $[E(x)] \in \{0, 1, 2\}$. We consider the 3 cases.

Case 1: $|\sqrt{x-1} - 1| = 0$ We get

$$\sqrt{x-1} - 1 = 0 \Rightarrow \sqrt{x-1} = 1 \Rightarrow x = 2.$$

Checking, we see that $[E(2)] = \left[\frac{4+4+3}{6}\right] = \left[\frac{11}{6}\right] = 1 \neq 0$. Thus, this case yields no solutions.

Case 2: $|\sqrt{x-1} - 1| = 1$ We get

$$\sqrt{x-1} - 1 = 1 \Rightarrow \sqrt{x-1} = 2 \Rightarrow x = 5$$

OR

$$\sqrt{x-1} - 1 = -1 \Rightarrow \sqrt{x-1} = 0 \Rightarrow x = 1$$

Checking, we see that $[E(5)] = \left[\frac{25+10+3}{27}\right] = \left[\frac{38}{27}\right] = 1 = |\sqrt{x-1} - 1|$. Thus, $x = 5$ is a solution.

On the other hand, $[E(1)] = \left[\frac{1+2+3}{3}\right] = \left[\frac{6}{3}\right] = 2 \neq 1$. Thus, $x = 1$ is false.

Case 3: $|\sqrt{x-1} - 1| = 2$ We get

$$\sqrt{x-1} - 1 = 2 \Rightarrow \sqrt{x-1} = 3 \Rightarrow x = 10$$

OR

$$\sqrt{x-1} - 1 = -2 \Rightarrow \sqrt{x-1} = -1 \text{ (no solution)}$$

Checking, we see that $[E(10)] = \left[\frac{100+20+3}{102}\right] = \left[\frac{123}{102}\right] = 1 \neq 2$. Thus, this case yields no solutions.

To conclude, the only solution is $x = 5$.

Problem 93A. Proposed by Luca Vlad Andrei

If $a, b, c, d \in [0, 1]$, prove that :

$$\frac{a+b+c}{8085 + b^{2022} + c^{2022} + d^{2022}} + \frac{a+b+d}{8085 + a^{2022} + b^{2022} + c^{2022}} +$$

$$+\frac{a+c+d}{8085 + a^{2022} + b^{2022} + d^{2022}} + \frac{b+c+d}{8085 + a^{2022} + c^{2022} + d^{2022}} \leq \frac{3}{2022}$$

Problem 94A. Proposed by Teodor Stupariu

Find $a, b, c > 0$ satisfying both conditions:

$$a^2 + b^2 + c^2 > 0$$

and

$$(16a^3+256a-68)(2b^3+8b+31)(6c^3+54c+162) \geq 9^3(a^2+94)(b^2+10)(c^2+45).$$

Problem 95A. Proposed by Matei Neacsu

Solve in $x, y, z \in \mathbb{R}^*$ the system: $\begin{cases} x+11 = \dfrac{y-25}{x} \\ y+11 = \dfrac{z-25}{y} \\ z+11 = \dfrac{x-25}{z} \end{cases}$

Solution Problem 95A

$\begin{cases} x+11 = \dfrac{y-25}{x} \\ y+11 = \dfrac{z-25}{y} \\ z+11 = \dfrac{x-25}{z} \end{cases} \Leftrightarrow \begin{cases} x^2+11x = y-25 \\ y^2+11y = z-25 \\ z^2+11z = x-25 \end{cases} \Rightarrow x^2+y^2+z^2+11(x+y+z) =$
$(x+y+z) - 75 \Rightarrow$
$= x^2+y^2+z^2+10(x+y+z)+75 = 0 \Leftrightarrow (x+5)^2+(y+5)^2+(z+5)^2 = 0 \Rightarrow$
$\begin{cases} x = -5 \\ y = -5 \\ z = -5 \end{cases}$
Conclusion: The solution is $x = y = z = -5$

Problem 96A. Proposed by Iancu-Ioan Sandea

Find the positive real numbers x, y, z satisfying both conditions: $x+y+z = 21$ and $\sqrt{21x+yz} + \sqrt{21y+xz} + \sqrt{21z+xy} \geq 42$.

Solution Problem 96A

We begin by proving that $\sqrt{21x+yz} + \sqrt{21y+xz} + \sqrt{21z+xy} \leq 42$.

Since $x+y+z = 21$, we have that

$$\begin{aligned} 21x + yz &= (x+y+z)x + yz \\ &= x^2 + xy + xz + yz \\ &= x(x+y) + z(x+y) \\ &= (x+z)(x+y). \end{aligned}$$

Thus, $\sqrt{21x+yz} = \sqrt{(x+z)(x+y)}$.

From AM-GM, we have that

$$\sqrt{(x+z)(x+y)} \le \frac{(x+z)+(x+y)}{2} \Rightarrow \sqrt{21x+yz} \le \frac{2x+y+z}{2}.$$

Similarly, $\sqrt{21y+xz} \le \frac{x+2y+z}{2}$ and $\sqrt{21z+xy} \le \frac{x+y+2z}{2}$.

Adding the above three inequalities, we obtain

$$\sqrt{21x+yz} + \sqrt{21y+xz} + \sqrt{21z+xy} \le \frac{4(x+y+z)}{2} = 2(x+y+z) = 42.$$

We can thus conclude that $\sqrt{21x+yz} + \sqrt{21y+xz} + \sqrt{21z+xy} = 42$.

Since we used AM-GM, equality occurs when

$$x+y = x+z, x+y = y+z, x+z = y+z \Leftrightarrow x = y = z.$$

Since $x+y+z = 21$, we conclude that $x = y = z = 7$.

Problem 97A. Proposed by Nicholas Edmond Teler

Find the last digit of the number $2^{2x} + 2^{2y}$ such that $4 \mid y$ and $2^{4x-5} + 2^{4y+3} \le 2^{2x+2y}$.

Solution Problem 97A

Using exponent laws, we rewrite the inequality as:

$$\frac{2^{4x}}{2^5} + 2^{4y} \cdot 2^3 \le 2^{2x} \cdot 2^{2y}.$$

Multiplying both sides by 2^5 to remove the denominator, we get

$$2^{4x} + 2^{4y} \cdot 2^8 \le 2^{2x} \cdot 2^{2y} \cdot 2^5$$

$$\Updownarrow$$

$$4^{2x} + 4^{2y} \cdot 16^2 - 4^x \cdot 4^y \cdot 2^5 \le 0$$

$$\Updownarrow$$

$$(4^x)^2 - 2 \cdot (4^x \cdot 4^y \cdot 16) + (4^y \cdot 16)^2 \le 0$$

$$\Updownarrow$$

$$(4^x - 4^y \cdot 16)^2 \le 0.$$

We know by Trivial Inequality that $(4^x - 4^y \cdot 16)^2 \ge 0$. Thus, we conclude that $(4^x - 4^y \cdot 16)^2 = 0$.

We now have that $4^x - 4^y \cdot 16 = 0 \Rightarrow 4^x = 4^y \cdot 16$.

Our desired expression now becomes

$$\begin{aligned} 2^{2x} + 2^{2y} &= 4^y \cdot 16 + 4^y \\ &= 4^y(16+1) \\ &= 4^y \cdot 17. \end{aligned}$$

Let $u(x)$ represent the value of the units digit of some number x.

Since $4 \mid y$, we know that the units digit of 4^y is the same as that of 4^4. Thus, $u(4^y) = u(4^4) = 6$. We also know that $u(17) = 7$.

Therefore, $u(4^y \cdot 17) = u(6 \cdot 7) = u(42) = 2$.

Problem 98A. Proposed by Teodor Melega

Given H the orthocenter of the triangle ABC and D the intersection of the tangent in B to the circle circumscribed to the triangle ABC with the line AH. The line DB intersects the second time the circle circumscribed to the triangle ABC in X. The line DC intersects the second time the circle circumscribed to the triangle ABC in Y. Prove that the intersection of XY and AB is on the circle circumscribed to the triangle AHC.

Solution Problem 98A

We assume that $\triangle ABC$ is acute and that $AB < AC$. The other cases can be proved analogously.

Let $XY \cap AB = \{Z\}$.

From Power of a Point with the circumcircles of triangles AHB and AHC, we have that

$$DB \cdot DX = DH \cdot DA = DY \cdot DC.$$

Since $DB \cdot DX = DY \cdot DC$, we see that $XBYC$ is cyclic.

Angle chasing, we have that

$$\begin{aligned} \measuredangle ZYH &= \measuredangle XYH \\ &= \measuredangle HYC - \measuredangle XYC \\ &= (180^\circ - \measuredangle HAC) - \measuredangle XBC \\ &= (180^\circ - (90^\circ - \measuredangle ACB)) - \measuredangle XBC && (\triangle \text{ HAC}) \\ &= (90^\circ + \measuredangle ACB) - (180^\circ - \measuredangle DBC) && \text{(supplementary)} \\ &= (90^\circ + \measuredangle ACB) - (180^\circ - \measuredangle BAC) && \text{(inscribed angles)} \\ &= 90^\circ - \measuredangle ABC \\ &= \measuredangle BAH \\ &= \measuredangle ZAH. \end{aligned}$$

Therefore, $AZHY$ is cyclic. Since three points define a circle and A, H, Y all lie on the circumcircle of $\triangle AHC$, we conclude that point Z will also lie on the circumcircle of $\triangle AHC$.

Problem 99A. Proposed by Iancu-Ioan Sandea

Given positive real numbers a, b, and c satisfying the condition $a + b + c = 15$. Find the minimum of the expression $E = 100(\frac{a}{15-a} + \frac{b}{15-b} + \frac{c}{15-c}) - a^2 - b^2 - c^2 + 2581$ and find the values of a, b, and c satisfying this minimum.

Solution Problem 99A

By replacing 15 with $a + b + c$ we obtain:
$E = 100(\frac{a}{a+b+c-a} + \frac{b}{a+b+c-b} + \frac{c}{a+b+c-c}) - a^2 - b^2 - c^2 + 2581 \iff$
$E = 100(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}) - a^2 - b^2 - c^2 + 2581.$
$a + b + c = 15 \iff a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 225 \implies$
$a^2 + b^2 + c^2 = 225 - 2(ab + bc + ac).$
By replacing again:
$E = 100(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}) - [225 - 2(ab + bc + ac)] + 2581 \iff$
$E = 100(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}) - 225 + 2(ab + bc + ac) + 2581 \iff$
$E = 100(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}) + 2(ab + bc + ac) + 2356.$
Using $AM \geq GM$: $\implies$
$\implies \frac{100a}{b+c} + ab + ac = \frac{100a}{b+c} + a(b + c) \geq \sqrt{\frac{100a}{b+c} \cdot a(b + c)} = \sqrt{100a^2} = 10a$ (because $a > 0$)
Similarly, $\frac{100b}{a+c} + ab + bc \geq 10b$ and $\frac{100c}{a+b} + ac + bc \geq 10c$.
By adding the three relationships :
$100(\frac{a}{15-a} + \frac{b}{15-b} + \frac{c}{15-c}) + 2(ab + bc + ac) \geq 10a + 10b + 10c \iff$
$100(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+c}) - a^2 - b^2 - c^2 + 2581 \geq 10(a+b+c) + 2356 = 10 \cdot 15 + 2356 = 150 + 2356 = 2506 \implies E \geq 2506.$
The problem is asking the minimum value for E that is **2506**.
This is obtained also for the equality (we used only AM-GM and the equality is obtained when the numbers are equal).
Thus $\frac{100a}{b+c} = a(b + c) \iff 100 = (b + c)^2 \iff b + c = 10$ (due to $b, c > 0$).
$\frac{100b}{a+c} = b(a + c) \iff 100 = (a + c)^2 \iff a + c = 10$ (due to $a, c > 0$).
$\frac{100c}{a+b} = c(a + b) \iff 100 = (a + b)^2 \iff a + b = 10$ (due to $a, b > 0$).
Given $a + b + c = 15 \implies$ **a=b=c=5**.

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 56B. Proposed by Alexander Monteith-Pistor

A game is played with white and black pieces and a chessboard (8 by 8). There is an unlimited number of identical black pieces and identical white pieces. To obtain a starting position, any number of black pieces are placed on one half of the board and any number of white pieces are placed on the other half (at most one piece per square). A piece is called matched if its color is the same of the square it is on. If a piece is not matched then it is mismatched. How many starting positions satisfy the following condition

$$\text{\# of matched pieces} - \text{\# of mismatched pieces} = 16$$

(your answer should be a binomial coefficient)

Problem 67B. Proposed by Stefan-Ionel Dumitrescu

Consider a cube $ABCDA'B'C'D'$. Point X lies on face $ADD'A'$. Point Y lies on face $ABB'A'$. Point W is a randomly chosen point on the edges, faces, or interior of the cube. If Z is the midpoint of XY, find the probability that W is the midpoint of an AZ.

Problem 76B. Proposed by Alexander Monteith-Pistor

Let $ABCD$ be a quadrilateral with $\measuredangle ABC = 90^\circ$. Points E and F are on AD and BC respectively such that AB is parallel to EF. Further, AC, BD and EF intersect at O. Given that $BF = 4$, $AB = 9$, $AE = 5$ and $CD = 20$, find a polynomial $p(x)$ such that one of its roots is at $x = \frac{DO}{OB}$.

Problem 77B. Proposed by Andy Kim

(i) Evaluate

$$\binom{n}{0} - 2\binom{n}{1} + \cdots \pm 2^n\binom{n}{n} = \sum_{i=0}^{n}(-1)^i 2^i \binom{n}{i}$$

for $n \in \mathbb{Z}_+$.
(ii) Prove that

$$\sum_{i=0}^{n}(-1)^{n-i} i^n \binom{n}{i} = n!$$

for all $n \in \mathbb{Z}_+$.

Problem 78B. Proposed by Ciurea Pavel

Given the positive real numbers x, y, and z, prove that

$$2(\sum_{cyc} x)\sqrt{\sum_{cyc}\sqrt{x^2+y^2+z^2}} \geq$$

$$\geq \sum_{cyc}\sqrt{3(x+y)(x+z)(\sqrt{x^2+y^2+xy}+\sqrt{x^2+z^2+xz}-\sqrt{y^2+z^2+yz})}.$$

Problem 80B. Proposed by Pavel Ciurea

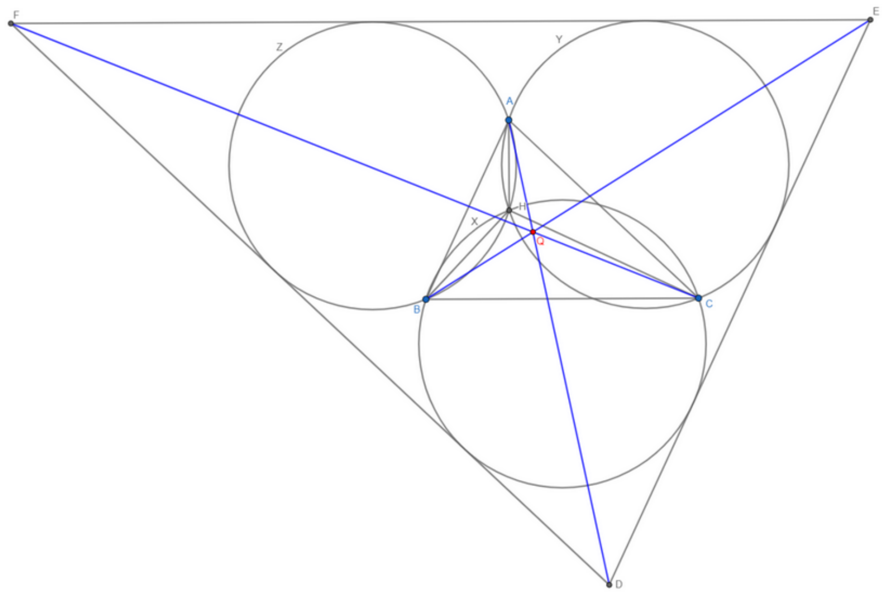
Given the positive real numbers x, y, z and t, prove that:

$$\frac{2xy}{yt}+\frac{2yt}{xz}+\frac{y}{z}+\frac{z}{y}+2\sqrt{(\frac{x}{y}+\frac{y}{x})(\frac{z}{t}+\frac{t}{z})(\frac{y}{z}+\frac{z}{y}-1)(\frac{x}{t}+\frac{t}{x}+1)} \geq \frac{x}{t}+\frac{t}{x}+\sqrt{3}(\frac{x}{z}+\frac{z}{x}+\frac{y}{t}+\frac{t}{y})+4.$$

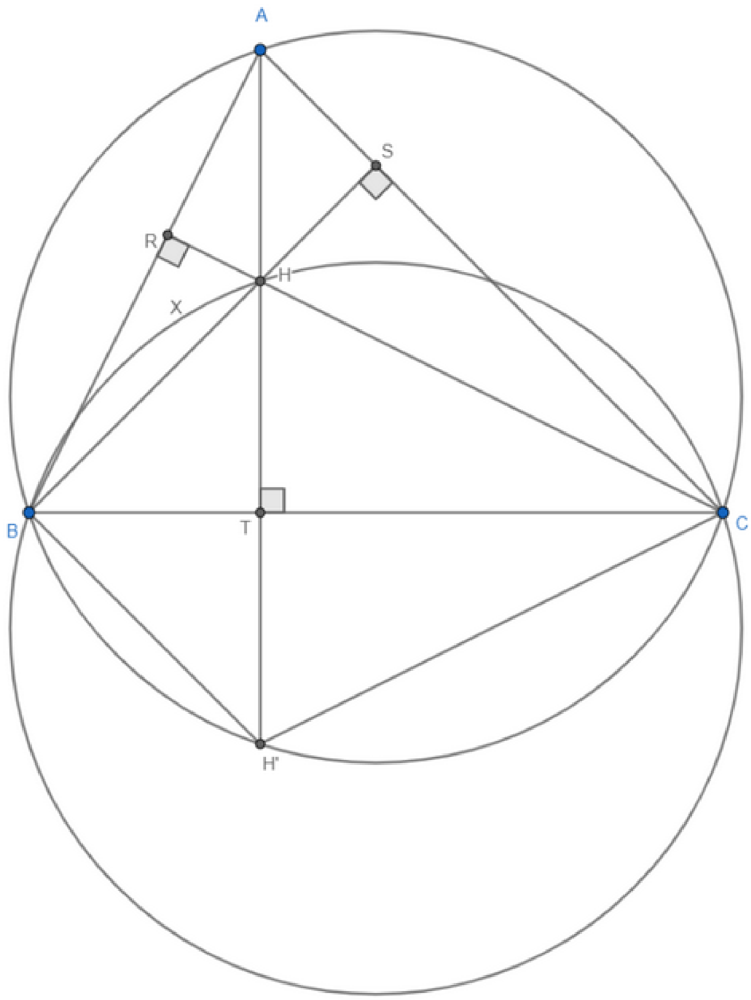
Problem 81B. Proposed by Alexandru Benescu

Let ABC be a triangle, H its orthocenter and X, Y, Z the circumscribed circles of ΔBHC, ΔAHC, and ΔAHB respectively. Let DE be the common tangent to X and Y, EF to Y and Z, and FD to X and Z, such that all three circles X, Y, Z are inside ΔDEF. Prove that AD, BE and CF are concurrent.

Solution Problem 81B



We first prove that circles X, Y, and Z are congruent.



Let H' be the reflection of point H over BC, W be the circumcircle of $\triangle ABC$, and α equal $\measuredangle BAC$. We also set R, T, S as the endpoints of the altitudes, as seen in the above diagram.

We see from cyclic quadrilaterals and opposite angles that $\measuredangle RHS = \measuredangle BHC = 180° - \alpha$. Because $\triangle BHC \equiv \triangle BH'C$, we have that

$$\measuredangle BH'C = 180° - \alpha \Rightarrow \measuredangle BAC + \measuredangle BH'C = 180°.$$

Thus, $ABH'C$ is inscribed, meaning that $H' \in W$. So, W is also the circumcircle of $\triangle BH'C$. Since X is the circumcircle of $\triangle BHC$, we know that X and W are congruent.

Similarly, Y and W are congruent and Z and W are congruent $\Rightarrow X, Y, Z$ are congruent.

Let O_Y be the center of Y and O_Z be the center of Z. Since $O_YA = AO_Z = O_ZH = HO_Y =$ radius, we conclude that O_YAO_ZH is a rhombus.

So, the diagonal property of a rhombus gives us that $AH \perp O_YO_Z$. Because $O_YO_Z || EF$ due to the property of radii and tangents, we know that $AH \perp EF$. As H is the orthocenter, we have that $AH \perp BC$. So, $BC||EF$.

Similarly, $AB||DE$ and $AC||DF$. Thus, $\triangle ABC \sim \triangle DEF \Rightarrow \frac{AB}{DE} = \frac{AC}{DF}$.

Let $\{Q\}$ be $AD \cap BA$. We will prove that C, Q, F are collinear.

Since $AB||DE \Rightarrow \triangle ABQ \sim \triangle DEQ \Rightarrow \frac{AQ}{DQ} = \frac{AB}{DE}$. Thus, $\frac{AQ}{DQ} = \frac{AC}{DF}$.

Combining this ratio with the equal alternate interior angles that $AC||DF$ gives, we have that $\triangle ACQ \sim \triangle DFQ$. Therefore,

$$\measuredangle AQC = \measuredangle DQF \Rightarrow \measuredangle AQD = \measuredangle CQF.$$

Since A, D, Q are collinear, we have that C, Q, F are collinear $\Rightarrow AD, BE$, and CF are concurrent.

Problem 82B. Proposed by Alexandru Benescu

Prove that:

$$(a+b+c+2)^2 + \frac{5}{2}(a+b)(b+c)(c+a) + 2(a^3+2)(b^3+2)(c^3+2) + 1 \geq 100abc$$

where a, b, $c \in \mathbb{R}_+$ and $a^2+b^2+c^2 = 3$.

Solution Problem 82B

From AM-GM,

$$a^3 + 2 = a^3 + 1 + 1 \geq 3\sqrt[3]{a^3 \cdot 1 \cdot 1} = 3a.$$

So, $2(a^3+2)(b^3+2)(c^3+2) \geq 2 \cdot 3a \cdot 3b \cdot 3c = 54abc$.

From AM-GM, we also know that $a + b \geq 2\sqrt{ab}$. Thus

$$(a+b)(b+c)(c+a) \geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} = 8abc.$$

$$\Downarrow$$

$$\frac{5}{2}(a+b)(b+c)(c+a) \geq 20abc.$$

Upon expanding and rearranging $(a+b+c+2)^2 + 1$, we have that

$$\begin{aligned}(a+b+c+2)^2 + 1 &= a^2+b^2+c^2+2ab+2bc+2ca+4a+4b+4c+5\\ &= 2ab+2bc+2ca+4a+4b+4c+8\\ &= 2abc \cdot \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}+\frac{4}{abc}\right).\end{aligned}$$

We will now write the above RHS as an inequality involving solely an abc term.

Since $a^2+b^2+c^2=3$, we know by AM-GM that $ab+bc+ca \le 3$.

Via Cauchy-Schwarz Inequality, we have that $(ab+bc+ca)(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}) \ge 9$. Therefore,

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \ge 3 \Rightarrow \frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca} \ge 6.$$

In addition, we see from $ab+bc+ca \le 3$ that

$$a^2+b^2+c^2+2ab+2bc+2ca \le 9 \Rightarrow (a+b+c)^2 \le 9 \Rightarrow a+b+c \le 3.$$

Via Cauchy-Schwarz Inequality, is known that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$. So, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3$.

Since we proved above that $a+b+c \le 3$, we know that $\frac{a+b+c}{3} \le 1$. Through AM-GM, we have $\sqrt[3]{abc} \le 1 \Rightarrow abc \le 1 \Rightarrow \frac{4}{abc} \ge 4$.

So,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}+\frac{4}{abc} \ge 3+6+4=13.$$

$$\Downarrow$$

$$(a+b+c+2)^2+1 \ge 13 \cdot 2abc = 26abc.$$

From above, we know that

$$\frac{5}{2}(a+b)(b+c)(c+a) \ge 20abc$$

and

$$2(a^3+2)(b^3+2)(c^3+2) \ge 54abc.$$

Combining everything together, we get that

$$(a+b+c+2)^2+\frac{5}{2}(a+b)(b+c)(c+a)+2(a^3+2)(b^3+2)(c^3+2)+1$$

$$\ge$$

$$abc(26+20+54)=100abc.$$

In conclusion, we have proved that

$$(a+b+c+2)^2+\frac{5}{2}(a+b)(b+c)(c+a)+2(a^3+2)(b^3+2)(c^3+2)+1 \ge 100abc.$$

Problem 86B. Proposed by Nicholas Sullivan

Suppose there are 2021 people sitting at a very large circular table. How many ways are there to give each person a red, blue or green hat such that no two neighbouring people have the same colour hat?

Solution Problem 86B

Let us consider this problem, but with n people sitting in a line. Let s_n denote the number of hat configurations that start and end with a blue hat. Let d_n denote the number of hat configurations that start with a blue hat and end with a red hat. By symmetry of hat colours, we know that d_n must be the number of configurations for any given beginning and ending colours that are different, and that s_n must be the number of configurations for any given beginning and ending colours that are the same.

If we have n people in a row, and the starting and ending hats are both blue, then the second hat can either be red or green. Thus, this is the same problem as having $n-1$ people in a row, starting with a red or green hat and ending with a blue hat. As a result:

$$s_n = 2d_{n-1}.$$

Similarly, if the starting hat is red and the ending hat is blue, then the second hat can either be blue or green. Thus, this is equivalent to having $n-1$ people in a row, starting with a blue or green hat and ending with a blue hat. As a result:

$$d_n = s_{n-1} + d_{n-1}.$$

We can combine these two relations to give the recurrence relation for s_n:

$$\begin{aligned} s_n &= 2d_{n-1} \\ &= 2(d_{n-2} + s_{n-2}) \\ &= s_{n-1} + 2s_{n-2}. \end{aligned}$$

The characteristic polynomial is thus $p(\lambda) = \lambda^2 - \lambda - 2$, which has roots $\lambda_1 = 2$ and $\lambda_2 = -1$. Since these roots are distinct, the general solution to the recurrence relation is:

$$s_n = a2^n + b(-1)^n.$$

We know that if we have 2 people, the starting and ending hats are next to each other, so they cannot be the same colour. Thus, $s_2 = 0$. Similarly, if we have 3 people, the middle hat can either be red or green, so $s_3 = 2$. Thus, $4a + b = 0$ and $8a - b = 2$, implying that $a = 1/6$, and $b = -2/3$. As a result, there are

$$s_n = \frac{2^n - 4(-1)^n}{6}$$

hat configurations for a line of n people that begin and end with a given color (e.g. blue).

Now, returning to the original problem, if we begin with a circular table of n people, then we can see that this is equivalent (for our purposes) to a line of

$n+1$ people, where the beginning and ending hat colours are the same. Thus, there are

$$\begin{aligned} c_n &= 3s_{n+1} \\ &= 3\frac{2^{n+1} - 4(-1)^{n+1}}{6} \\ &= 2^n + 2(-1)^n \end{aligned}$$

ways to distribute hats in this way. Thus, for $n = 2021$, there are:

$$c_{2021} = 2^{2021} - 2$$

hat configurations.

Problem 87B. Proposed by Max Jiang

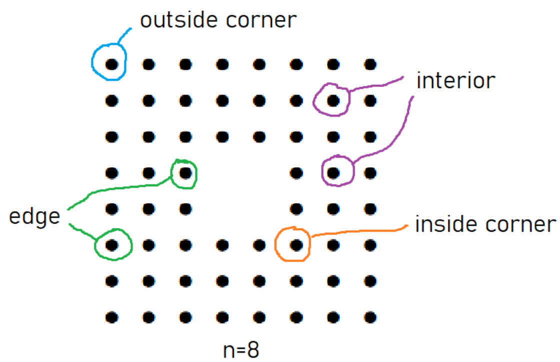
There is a row of n pies, numbered 1 to n from left to right. You start at pie 1 and go rightward. At each pie, there is a $1/2$ chance that you eat the pie, after which you move on to the next uneaten pie (which depends on the direction you are moving). Upon reaching the last uneaten pie, you change the direction you are going (if the last pie still uneaten you will "repeat" it). You continue until all pies are eaten. What is the probability that pie 1 is the last pie you eat? Express your answer as a finite sum in terms of n.

Problem 88B. Proposed by Daisy Sheng

Pierre is coloring a $n \times n$ square grid, where n is even and $n \geq 8$. He chooses to omit the centre 2×2 square grid (see diagram for an example). In the spirit of the holidays, Pierre is coloring the dots either red or green. He also connects horizontally, vertically, and diagonally adjacent dots with lines using the following color scheme rules:

- 2 red dots are connected by a gold line segment.
- 2 green dots are connected by a blue line segment.
- A red and green dot are connected by a silver line segment.

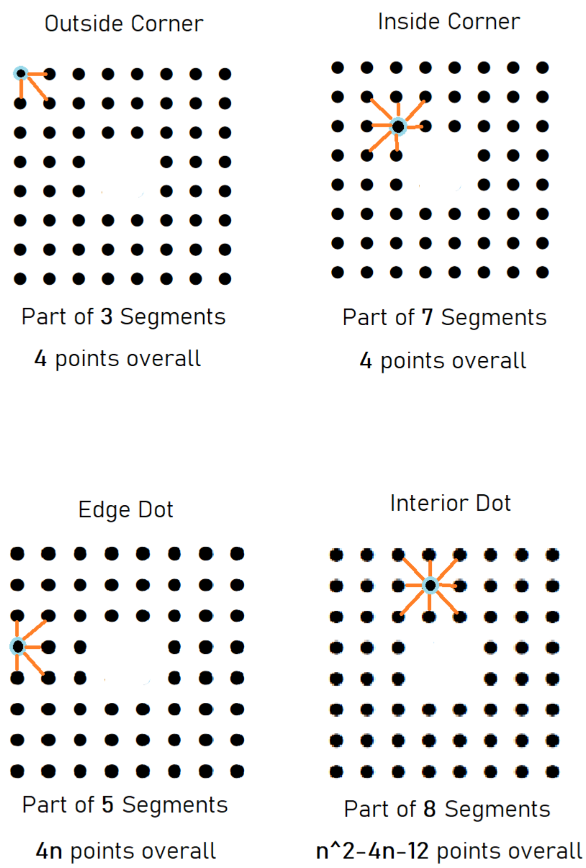
Let there be: r_0 red outside corner dots, r_1 red inside corner dots, r_2 red edge dots, and r_3 red interior dots. If B is the overall number of blue line segments and G is the overall number of gold line segments, find G in terms of B, n, r_0, r_1, r_2, and r_3.



Solution Problem 88B

We see that Pierre's $n \times n$ grid of dots has 4 outside corners, 4 inside corners, and $4(n-2)+4(2) = 4n-8+8 = 4n$ edge dots. Since there are n^2-4 points in total, we see that there are $(n^2-4)-4-4-4n = n^2-4n-12$ interior points.

We take a look at how many line segments each type of point connects to.



Adding up all the line segments corresponding with each dot, we see that each line segment is double counted since endpoints are being considered. Letting S be the number of silver line segments, we have that

$$\begin{aligned} 2B + 2G + 2S &= 3 \cdot 4 + 7 \cdot 4 + 5 \cdot 4n + 8 \cdot (n^2 - 4n - 12) \\ &= 12 + 28 + 20n + 8n^2 - 32n - 96 \\ &= 8n^2 - 12n - 56. \end{aligned}$$

Thus,

$$B + G + S = 4n^2 - 6n - 28. \tag{1}$$

Adding up all the line segments corresponding to red dots, we see that gold line segments are double counted while silver line segments are counted only once.

We get that

$$2G + S = 3 \cdot r_0 + 7 \cdot r_1 + 5 \cdot r_2 + 8 \cdot r_3. \qquad (2)$$

Subtracting equation 1 from equation 2, we get

$$G - B = 3 \cdot r_0 + 7 \cdot r_1 + 5 \cdot r_2 + 8 \cdot r_3 - 4n^2 - 6n - 28,$$

which gives us that $G = B + 3r_0 + 7r_1 + 5r_2 + 8r_3 - 4n^2 - 6n - 28$.

Problem 89B. Proposed by Arnab Sanyal

Let ω_1 and ω_2 be two intersecting circles. Let $\omega_1 \cap \omega_2 = \{P, Q\}$. Let the tangent to ω_1 and ω_2 at P meet ω_1 and ω_2 respectively at B and A. Let the circumcircle of ΔPBA be ω_0. AQ meets ω_0 (possibly extended) at Y and BQ meets ω_0 (possibly extended) at X. Assuming $P \neq Q$, prove or disprove the following statements:
(i) $XABY$ is an isosceles trapezoid;
(ii) $QOBA$ is cyclic, where O is the circumcenter of ω_0.

Problem 90B. Proposed by Gabriel Crisan

If $a, b, c \in \mathbb{R}$ prove that :

$$\frac{1}{2^{a-b+3}+1} + \frac{1}{2^{b-c+3}+1} + \frac{1}{2^{c-a+3}+1} \geq \frac{1}{3}$$

Solution Problem 90B

We can rewrite the inequality as

$$\frac{1}{8 \cdot 2^{a-b}+1} + \frac{1}{8 \cdot 2^{b-c}+1} + \frac{1}{8 \cdot 2^{c-a}+1} \geq \frac{1}{3}.$$

We observe that $(2^{a-b}) \cdot (2^{b-c}) \cdot (2^{c-a}) = 1$. This means that we can set $2^{a-b} = \frac{x}{y}$, $2^{b-c} = \frac{y}{z}$, and $2^{c-a} = \frac{z}{x}$, where $x, y, z \neq 0$.

Our inequality can thus be written as

$$\frac{1}{1+\frac{8x}{y}} + \frac{1}{1+\frac{8y}{z}} + \frac{1}{1+\frac{8z}{x}} \geq \frac{1}{3} \Leftrightarrow \frac{y}{8x+y} + \frac{z}{8y+z} + \frac{x}{8z+x} \geq \frac{1}{3}.$$

We know that for any $m, n, p \in \mathbb{R}$ and $q, r, s \in \mathbb{R}^+$, Cauchy-Schwarz yields the following true inequality:

$$\frac{m^2}{q} + \frac{n^2}{r} + \frac{p^2}{s} \geq \frac{(m+n+p)^2}{q+r+s}.$$

Multiplying the top and bottom of each fraction with its numerator and applying the above formula, we get that

$$\frac{y^2}{8xy+y^2}+\frac{z^2}{8yz+z^2}+\frac{x^2}{8xz+x^2}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2+8xy+8yz+8xz}.$$

This means that if we prove that $\frac{(x+y+z)^2}{x^2+y^2+z^2+8xy+8yz+8xz}\geq\frac{1}{3}$, the problem would be solved.

The inequality is equivalent with:

$$3(x+y+z)^2\geq x^2+y^2+z^2+8xy+8yz+8xz$$

$$\Downarrow$$

$$3x^2+3y^2+3z^2+6xy+6yz+6xz\geq x^2+y^2+z^2+8xy+8yz+8zx$$

$$\Downarrow$$

$$2x^2+2y^2+2z^2\geq 2xy+2yz+2zx$$

$$\Downarrow$$

$$2x^2+2y^2+2z^2-2xy-2yz-2zx\geq 0$$

$$\Downarrow$$

$$(x-y)^2+(y-z)^2+(x-z)^2\geq 0,$$

which is true by the Trivial Inequality.

Thus,

$$\frac{y^2}{8xy+y^2}+\frac{z^2}{8yz+z^2}+\frac{x^2}{8xz+x^2}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2+8xy+8yz+8xz}\geq\frac{1}{3}.$$

With the above, we conclude that

$$\frac{1}{2^{a-b+3}+1}+\frac{1}{2^{b-c+3}+1}+\frac{1}{2^{c-a+3}+1}\geq\frac{1}{3}.$$

Problem 91B. Proposed by Stefan Ionel Dumitrescu

Let $n\geq 3$ and $m\geq 2$ natural numbers, with $n>m$. Let's define a table of real numbers with m lines and n columns:

$$T=\begin{pmatrix} a_{11} & \cdots & a_{n1}\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$$

having the property that the sum of squares of the numbers inscribed in each column is 1, namely:

$$a_{1i}^2+\cdots+a_{mi}^2=1,(\forall)\ i\in\overline{1,n}$$

For each line, we define a variable A_j that 'measures' the sum of all the products of 2 numbers taken from all n numbers that appear on that line, meaning that:

$$A_j = a_{j1}a_{j2} + a_{j1}a_{j3} + \cdots + a_{j(n-1)}a_{jn}, (\forall)\ j \in \overline{1,m}$$

Prove that:

$$\sum_{i=1}^{m} A_i < \frac{n^n}{4(n-2)^{n-2}}$$

Solution Problem 91B

Lemma. Let p, q be natural numbers, with $2 \leq q < p$. Then, it is true that:

$$\binom{p}{q} < \frac{p^p}{q^q(p-q)^{p-q}}.$$

Proof.

The inequality can be rewritten as:

$$p^p > \binom{p}{q} q^q (p-q)^{p-q}.$$

Since $q < p$, we can let $r = p - q$. By using the binomial expansion formula, we have that

$$p^p = (q+r)^p = \sum_{i=0}^{p} \binom{p}{i} q^i r^{p-i} = \sum_{i=0}^{p} \binom{p}{i} q^i (p-q)^{p-i}.$$

We thus have to prove that:

$$\sum_{i=0}^{p} \binom{p}{i} q^i (p-q)^{p-i} > \binom{p}{q} q^q (p-q)^{p-q}.$$

The inequality is obviously true, since a sum of $p+1$ terms is always greater than one of its terms (in this case, the term for $i = q$).

Getting back to the solution, we choose some points $P_i\,(a_{1i}, a_{2i}, \ldots, a_{mi})$ in the n-dimensional space. From the sum of squares condition, it is obvious that all such points are on the unit $(m-1)$-sphere centered at the origin, O, of the n-dimensional Cartesian coordinate system. Defining for each one a position vector, each having length 1, and using Schwarz's inequality for the scalar product of two vectors, we know:

$$\overrightarrow{OP_i} \cdot \overrightarrow{OP_j} \leq |\overrightarrow{OP_i}||\overrightarrow{OP_j}| = 1.$$

Expressing the dot product analytically, we can get:

$$a_{1i}a_{1j} + a_{2i}a_{2j} + \cdots + a_{mi}a_{mj} < 1.$$

There are $\binom{n}{2}$ pairs of numbers that i and j can be set to.

By summing all these distinct pairs of is and js, we can conclude that:

$$\sum_{i=1}^{m} A_i < \binom{n}{2} \cdot (1).$$

From the earlier lemma, we have that

$$\sum_{i=1}^{m} A_i < \binom{n}{2} < \frac{n^n}{4(n-2)^{n-2}}.$$

This completes the proof.

Problem 92B. Proposed by Gabriel Crisan

If $a, b, c \in \mathbb{R}^+$ and $\log_a b \cdot \log_b c \cdot \log_c a = 1$, prove that :

$$\frac{1}{\log_a ab^7} + \frac{1}{\log_b bc^7} + \frac{1}{\log_c ca^7} \geq \frac{3}{8}$$

Solution Problem 92B

We can rewrite the inequality as

$$\frac{1}{1 + 7\log_a b} + \frac{1}{1 + 7\log_b c} + \frac{1}{1 + 7\log_c a} \geq \frac{3}{8}.$$

Since we are given that $(\log_a b) \cdot (\log_b c) \cdot (\log_c a) = 1$, we can set $\log_a b = \frac{x}{y}$, $\log_b c = \frac{y}{z}$, and $\log_c a = \frac{z}{x}$, where $x, y, z \neq 0$.

Our inequality can thus be written as

$$\frac{1}{1 + \frac{7x}{y}} + \frac{1}{1 + \frac{7y}{z}} + \frac{1}{1 + \frac{7z}{x}} \geq \frac{3}{8} \Leftrightarrow \frac{y}{7x + y} + \frac{z}{7y + z} + \frac{x}{7z + x} \geq \frac{3}{8}.$$

We know that for any $m, n, p \in \mathbb{R}$ and $q, r, s \in \mathbb{R}^+$, the following inequality is true due to Cauchy-Schwarz:

$$\frac{m^2}{q} + \frac{n^2}{r} + \frac{p^2}{s} \geq \frac{(m + n + p)^2}{q + r + s}.$$

Multiplying the top and bottom of each fraction with its numerator and using the above formula on the left hand side of our obtained inequality, we have that

$$\frac{y^2}{7xy + y^2} + \frac{z^2}{7yz + z^2} + \frac{x^2}{7xz + x^2} \geq \frac{(x + y + z)^2}{x^2 + y^2 + z^2 + 7xy + 7yz + 7xz}.$$

This means that if we prove that $\frac{(x+y+z)^2}{x^2+y^2+z^2+7xy+7yz+7xz} \geq \frac{3}{8}$, the problem would be solved.

The inequality is equivalent with:

$$8(x+y+z)^2 \geq 3(x^2+y^2+z^2+7xy+7yz+7xz)$$

$$\Downarrow$$

$$8x^2+8y^2+8z^2+16xy+16yz+16xz \geq 3x^2+3y^2+3z^2+21xy+21yz+21zx$$

$$\Downarrow$$

$$5x^2+5y^2+5z^2 \geq 5xy+5yz+5zx$$

$$\Downarrow$$

$$2x^2+2y^2+2z^2-2xy-2yz-2zx \geq 0$$

$$\Downarrow$$

$$(x-y)^2+(y-z)^2+(x-z)^2 \geq 0,$$

which is true by the Trivial Inequality.

Thus,

$$\frac{y^2}{7xy+y^2}+\frac{z^2}{7yz+z^2}+\frac{x^2}{7xz+x^2} \geq \frac{3}{8},$$

meaning that $\frac{1}{\log_a ab^7}+\frac{1}{\log_b bc^7}+\frac{1}{\log_c ca^7} \geq \frac{3}{8}$.

Problem 93B. Proposed by Radu Stoleru

Given the triangle ABC with all angles less than 75 degrees. Find the point M in interior of the triangle such that $MA \cdot \frac{1+\sqrt{3}}{\sqrt{2}} + MC \cdot \sqrt{2} + MC$ is minimum.

Problem 94B. Proposed by Iulia Slanina

Let (x_n) be a sequence in $\mathbb{R}^+$ such that $x_1 = \frac{1}{16}$ and $x_{n+1} = 16x_n^3$. Let (y_n) be a sequence in $\mathbb{R}^+$ such that $y_n = 16x_n^2 + 4x_n + 1$.
Find $\lim_{n\to\infty}(y_1 y_2 \cdots y_n)$.

Solution Problem 94B

We can rewrite y_n as:

$$y_n = \frac{(4x_n-1)(16x_n^2+4x_n+1)}{4x_n-1} = \frac{64x_n^3-1}{4x_n-1} = \frac{4x_{n+1}-1}{4x_n-1}$$

Consequently the product $y_1 y_2 \cdots y_n$ will be:

$$y_1 y_2 \cdots y_n = \frac{4x_{n+1}-1}{4x_1-1} = \frac{4-16x_{n+1}}{3}.$$

Let (a_n) be a sequence in $\mathbb{R}$, where $a_n = \frac{1}{4x_n}$ and $a_1 = 4$. We have $x_n = \frac{1}{4a_n}$ and $x_{n+1} = 16x_n^3 = \frac{16}{64a_n^3} = \frac{1}{4a_n^3}$.
Consequently $a_{n+1} = a_n^3$.
By induction, $a_{n+1} = 4^{3^n}$ and $x_{n+1} = \frac{1}{4^{3^n+1}}$.
We can conclude that $\lim_{n\to\infty}(x_{n+1}) = 0$. From the above computations, we have

$$\lim_{n\to\infty}(y_1 y_2 \cdots y_n) = \lim_{n\to\infty}\frac{4-16x_{n+1}}{3} = \frac{4}{3}.$$

Problem 95B. Proposed by Daisy Sheng

Prove that

$$6n + \prod_{k=0}^{2n-1}\frac{k^3+2k^2+2k+1}{k^3-(2n+1)k^2+(2n+1)k-2n}$$

is an odd perfect square for all $n \in \mathbb{Z}^+$.

Solution Problem 95B

We have that $k^3+2k^2+2k+1 = (k+1)(k^2+k+1)$. Through factoring by grouping, the denominator in the product becomes

$$\begin{aligned} k^3-(2n+1)k^2+(2n+1)k-2n &= (k^3-2n\cdot k^2)-(k^2-2n\cdot k)+(k-2n), \\ &= k^2(k-2n)-k(k-2n)+(k-2n), \\ &= (k-2n)(k^2-k+1). \end{aligned}$$

Let our product be P. It can be re-written as

$$\begin{aligned} P &= \prod_{k=0}^{2n-1}\frac{(k+1)(k^2+k+1)}{(k-2n)(k^2-k+1)}, \\ &= \prod_{k=0}^{2n-1}\frac{(k+1)}{(k-2n)}\cdot\prod_{k=0}^{2n-1}\frac{(k^2+k+1)}{(k^2-k+1)}. \end{aligned}$$

We see that

$$\begin{aligned} \prod_{k=0}^{2n-1}\frac{(k+1)}{(k-2n)} &= \frac{1\cdot 2\cdot \ldots \cdot 2n}{(-2n)\cdot(1-2n)\cdot\ldots\cdot(-1)}, \\ &= \frac{1\cdot 2\cdot\ldots\cdot 2n}{(-1)^{2n}\cdot 1\cdot 2\cdot\ldots\cdot 2n}, \\ &= 1. \end{aligned}$$

Thus,

$$P = \prod_{k=0}^{2n-1}\frac{(k^2+k+1)}{(k^2-k+1)}.$$

We notice that $(k+1)^2-(k+1)+1=k^2+2k+1-k-1+1=k^2+k+1$. For example, $3^2-3+1=2^2+2+1=7$. So, many terms in the numerator will cancel with terms in the denominator.

Our product becomes

$$\begin{aligned}\prod_{k=0}^{2n-1}\frac{(k^2+k+1)}{(k^2-k+1)} &= \frac{1\cdot(1^2+1+1)\cdot(2^2+2+1)\cdot\ldots\cdot((2n-1)^2+(2n-1)+1)}{1\cdot(1^2-1+1)\cdot(2^2-2+1)\ldots\cdot((2n-1)^2-(2n-1)+1)},\\ &= \frac{(2n-1)^2+(2n-1)+1}{(1^2-1+1)},\\ &= (2n-1)^2+(2n-1)+1,\\ &= 4n^2-4n+1+2n-1+1,\\ &= 4n^2-2n+1.\end{aligned}$$

Our original expression is thus

$$\begin{aligned}6n+\prod_{k=0}^{2n-1}\frac{k^3+2k^2+2k+1}{k^3-(2n+1)k^2+(2n+1)k-2n} &= 6n+(4n^2-2n+1),\\ &= 4n^2+4n+1,\\ &= (2n+1)^2.\end{aligned}$$

Since $(2n+1)^2$ is an odd perfect square for all $n\in\mathbb{Z}^+$, the problem is proved.

Problem 96B. Proposed by Vedaant Srivastava

Arrange the numbers from 1 to 2022 in a circle. Label the numbers $(x_1,x_2,\ldots,x_{2022})$ moving clockwise around the circle. By convention, we define $x_i=x_j$ if $i\equiv j \pmod{2022}$. Define a function

$$f(\mathcal{P})=\sum_{i=1}^{2022}|x_i-x_{i+1}|$$

For some permutation $\mathcal{P}$ of $(x_1,x_2,\ldots,x_{2022})$. Let $f_{\max}$ and $f_{\min}$ be the maximimum and minimum values of f, respectively, over all permutations $\mathcal{P}$. Determine

$$f_{\max}-f_{\min}$$

Solution Problem 96B

First, we will calculate $f_{\min}$. Let j,k be such that $x_j=2022$ and $x_k=1$. Then by the triangle inequality, we have

$$\sum_{i=j}^{k-1}|x_i-x_{i+1}|\geq\left(|\sum_{i=j}^{k-1}(x_i-x_{i+1})\right)|=|x_j-x_k|=2021$$

In a similar manner, by the triangle inequality, we have

$$\sum_{i=k}^{j-1} |x_i - x_{i+1}| \geq \left(|\sum_{i=k}^{j-1}(x_i - x_{i+1}) \right) | = |x_k - x_j| = 2021$$

Thus

$$\sum_{i=1}^{2022} |x_i - x_{i+1}| = \sum_{i=j}^{j-1} |x_i - x_{i+1}| = \sum_{i=j}^{k-1} |x_i - x_{i+1}| + \sum_{i=k}^{j-1} |x_i - x_{i+1}| \geq 2021 + 2021 = 4042$$

And $f_{\min} = 4042$. This lower bound can be achieved with the permutation $\mathcal{P}_{\min} = (1, 2, 3, \dots 2022)$.

To calculate $f_{\max}$, the key observation is that $|x_i - x_{i+1}| = \max(x_i, x_{i+1}) - \min(x_i, x_{i+1})$. Then we obtain

$$\sum_{i=1}^{2022} |x_i - x_{i+1}| = \sum_{i=1}^{2022} \max(x_i, x_{i+1}) - \min(x_i, x_{i+1}) = \sum_{i=1}^{2022} \max(x_i, x_{i+1}) - \sum_{i=1}^{2022} \min(x_i, x_{i+1})$$

In each of the two sums written above, we have that each of the elements from 1 to 2022 can appear at most twice and that each of the sums contains exactly 2022 elements. This implies that

$$\sum_{i=1}^{2022} \max(x_i, x_{i+1}) \leq (2022+2022+2021+2021+2020+\cdots+1012+1012) = 1011 \cdot 3034$$

Similarly,

$$\sum_{i=1}^{2022} \min(x_i, x_{i+1}) \geq (1 + 1 + 2 + 2 + 3 + \cdots + 1011 + 1011) = 1011 \cdot 1012$$

Subtracting the second inequality from the first yields

$$\sum_{i=1}^{2022} \max(x_i, x_{i+1}) - \sum_{i=1}^{2022} \min(x_i, x_{i+1}) \leq 1011 \cdot 3034 - 1011 \cdot 1012 = 2044242$$

Thus $f_{\max} = 2044242$. This value can be achieved by a permutation $\mathcal{P}_{\max}$ where the even-indexed elements are described by the permutation $(1012, 1013, \dots, 2022)$ and the odd-indexed elements are described by the permutation $(1, 2, \dots, 1011)$.

Finally we have that

$$f_{\max} - f_{\min} = 2044242 - 4042 = \boxed{2040200}$$

Problem 97B. Proposed by Max Jiang

Anthony the Ant likes crawling on Platonic solids. He always starts on a vertex of a Platonic solid. Every second, he chooses one of the edges at the vertex uniformly at random and walks along it to the vertex on the other side. Find the probability that the first time he comes back to the first vertex is after an even number of seconds if he is crawling on a: (i) Tetrahedron, (ii) Cube, or (iii) Dodecahedron.

Problem 98B. Proposed by Daisy Sheng

Let c be an integer constant, where $c \geq 2$. Given that $x, y, w, z \in \mathbb{N}^+$ and

$$(c+1)(ywz + 2y + 101z) = c(xywz + 2xy + 101xz + wz + 2),$$

find all possible values of $x + y + w + z$ in terms of c.

Solution Problem 98B

We present the solution proposed by Arnab Sanyal.

We can rewrite the equation as

$$(c + 1 - cx)(ywz + 2y + 101z) = c(wz + 2).$$

Rearranging gives us

$$c + 1 - cx = \frac{c(wz+2)}{ywz + 2y + 101z} > 0$$

$$\Downarrow$$

$$c + 1 > cx$$

$$\Downarrow$$

$$x < 1 + \frac{1}{c} \leq 1 + \frac{1}{2} \qquad (\because c \geq 2)$$

Therefore, $x \leq 1$. Because $x \in \mathbb{N}$, we know that $x = 1$.

Substituting this information into the equation gives us

$$1 = \frac{c(wz+2)}{ywz + 2y + 101z}$$

$$\Downarrow$$

$$ywz + 2y + 101z = c(wz + 2)$$

$$\Downarrow$$

$$(wz + 2)(c - y) = 101z.$$

As $c, y \in \mathbb{N}$, we know that

$$(wz + 2) \mid (101z)$$

$$\Downarrow$$

$$(wz + 2) \mid (101wz)$$

$$\Downarrow$$

$$(wz + 2) \mid ((101wz) - 101 \cdot (wz + 2))$$

$$\Downarrow$$
$$(wz+2) \mid (-202).$$

Therefore, $(wz+2) \mid (202)$.

Also, $wz+2>2 \Rightarrow wz+2=101$ or 202. We now have two cases: $wz=99$ or $wz=200$.

Case 1: wz=99

Then, $101(c-y)=101z \Rightarrow y=c-z$.

As $w,z \in \mathbb{N}$, we know that $(w,z)=(1,99),(3,33),(9,11),(11,9),(33,3),(99,1)$.

So, $(x+y+z+w)=(1+c+1),(1+c+3),(1+c+9),(1+c+11),(1+c+33),(1+c+99)$. Therefore, this case gives us

$$x+y+z+w \in \{c+2,c+4,c+10,c+12,c+34,c+100\}.$$

Case 2: wz=200

Then, $202(c-y)=101z \Rightarrow z=2(c-y)$. So, z must be even.

As $w,z \in \mathbb{N}$ and z is even, we have that

$(w,z)=(1,200),(2,100),(5,40),(10,20),(4,50),(100,2),(20,10),(25,8),(50,4)$.

So, $x+y+z+w=1+\frac{2c-z}{2}+z-w=1+c+\frac{z}{2}+w$.

Therefore, this case gives us

$$x+y+z+w \in \{c+53,c+102,c+30,c+26,c+21\}.$$

To conclude, the possible values of $(x+y+w+z)$ are $\{c+2,c+4,c+10,c+12,c+34,c+100,c+53,c+102,c+30,c+26,c+21\}$.

Addition by the AEs: Alternative Method to Prove x=1.

The equation can be rearranged as

$$\begin{aligned}\frac{c+1}{c}=1+\frac{1}{c}&=\frac{xywz+2xy+101xz+wz+2}{ywz+2y+101z},\\ &=\frac{x(ywz+2y+101z)+wz+2}{ywz+2y+101z},\\ &=x+\frac{wz+2}{ywz+2y+101z}.\end{aligned}$$

Taking the reciprocal of the fraction, we have

$$\begin{aligned} x + \frac{wz+2}{ywz+2y+101z} &= x + \frac{1}{\frac{ywz+2y+101z}{wz+2}}, \\ &= x + \frac{1}{y + \frac{101z}{wz+2}}, \\ &= x + \frac{1}{y + 101 \cdot \frac{z}{wz+2}}, \\ &= x + \frac{1}{y + 101 \cdot \frac{1}{\frac{wz+2}{z}}}, \\ &= x + \frac{1}{y + \frac{101}{w+\frac{2}{z}}}. \end{aligned}$$

Thus, we have that

$$1 + \frac{1}{c} = x + \frac{1}{y + \frac{101}{w+\frac{2}{z}}}.$$

Since $x, y, w, z \in \mathbb{N}^+$, we know that $y + \frac{101}{w+\frac{2}{z}} > 1$. So, $\frac{1}{y+\frac{101}{w+\frac{2}{z}}} < 1$, meaning that $\boxed{x = 1}$.

Problem 99B. Proposed by Vedaant Srivastava

Arrange the numbers from 1 to 2022 in a circle. Label the numbers $(x_1, x_2, \ldots, x_{2022})$ moving clockwise around the circle. By convention, we define $x_i = x_j$ if $i \equiv j \pmod{2022}$. Define a function

$$g(\mathcal{P}) = \sum_{i=1}^{2022} x_i \cdot x_{i+1}$$

For each permutation $\mathcal{P}$ of $(x_1, x_2, \ldots, x_{2022})$. Determine the maximum of $g(\mathcal{P})$ over all permutations $\mathcal{P}$.

Solution Problem 99B

The key observation is to notice that

$$\sum_{i=1}^{2022} x_i x_{i+1} = \frac{1}{2}\left(\sum_{i=1}^{2022} x_i^2 + x_{i+1}^2 - \sum_{i=1}^{2022}(x_i - x_{i+1})^2\right) = \sum_{i=1}^{2022} i^2 - \frac{1}{2}\sum_{i=1}^{2022}(x_i - x_{i+1})^2$$

So to find the maximum of $\sum_{i=1}^{2022} x_i x_{i+1}$ it suffices to find the minimum of $\sum_{i=1}^{2022}(x_i - x_{i+1})^2$.

Let j, k be indices such that $x_j = 2022, x_k = 1$. Write $y_i = (x_i - x_{i+1})$ for $i \in [j, k-1]$. Write $y_i = (x_{i+1} - x_i)$ for $i \in [k, j-1]$. Then we have that

$$\sum_{i=1}^{2022} y_i = \sum_{i=j}^{k-1} y_i + \sum_{i=k}^{j-1} y_i = \sum_{i=j}^{k-1}(x_i - x_{i+1}) + \sum_{i=k}^{j-1}(x_{i+1} - x_i) = (x_j - x_k) + (x_j - x_k) = 4042$$

Thus, our goal is to find the minimum of

$$S = \sum_{i=1}^{2022}(x_i - x_{i+1})^2 = \sum_{i=1}^{2022} y_i^2$$

given the fixed sum $\sum_{i=1}^{2022} y_i = 4042$ for $y_i \in \mathbb{Z}$.

Now, for some helpful results. Given two integers a, b that have the same parity, we can obtain

$$\begin{aligned}
(a-b)^2 &\geq 0 \\
a^2 + b^2 &\geq 2ab \\
2(a^2 + b^2) &\geq a^2 + 2ab + b^2 \\
a^2 + b^2 &\geq \left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 \qquad (1)
\end{aligned}$$

Where equality holds iff $a = b$. Note that $\frac{a+b}{2}$ is an integer. Also, for integers a, b with opposite parity, we have

$$\begin{aligned}
(a-b)^2 &\geq 1 \\
a^2 + b^2 &\geq 1 + 2ab \\
a^2 + b^2 &\geq \frac{2a^2 + 2b^2 + 2 + 4ab}{4} \\
a^2 + b^2 &\geq \left(\frac{a+b+1}{2}\right)^2 + \left(\frac{a+b-1}{2}\right)^2 \qquad (2)
\end{aligned}$$

Where equality holds iff $|a - b| = 1$. Note that $\frac{a+b+1}{2}, \frac{a+b-1}{2}$ are integers.

Thus, if there exist indices r, s such that $|y_r - y_s| \geq 2$, we can replace y_r, y_s with $\frac{y_r+y_s}{2}, \frac{y_r+y_s}{2}$ or $\frac{y_r+y_s+1}{2}, \frac{y_r+y_s-1}{2}$ to yield a configuration with

$$\sum_{i=1}^{2022} y_i = 4042$$

and smaller S.

Thus any minimal set of $\{y_i\}$ must have $y_i \in \{a, a+1\}$ for some integer a. Note that the average of all y_i is

$$\overline{y} = \frac{1}{2022}\sum_{i=1}^{2022} y_i = \frac{4042}{2022}$$

As $y_i \in [a, a+1]$ implies $a < \overline{y} < a+1$, then we must have $a = 2$, so all y_i are either 1 or 2.

Then the only way such that $\sum_{i=1}^{2022} y_i = 4042$ is if there are 2020 2's and 2 1's among the y_i.

In fact, this configuration can be achieved by the permutation $\mathcal{P}_{\min} = (2022, 2020, 2018, \ldots, 2, 1, 3, 5, \ldots, 2021)$

Thus we have that

$$\sum_{i=1}^{2022}(x_i - x_{i+1})^2 = \sum_{i=1}^{2022} y_i^2 \geq 2 \cdot 1^2 + 2020 \cdot 2^2 = 8082$$

Then

$$\sum_{i=1}^{2022} x_i x_{i+1} = \sum_{i=1}^{2022} i^2 - \frac{1}{2}\sum_{i=1}^{2022}(x_i - x_{i+1})^2 \leq \frac{2022 \cdot 2023 \cdot 4045}{6} - \frac{1}{2} \cdot 8082 = \boxed{2757678754}$$