Number 4 - January 2021 Problems

February 22, 2021

Problems

Problem 27A. Proposed by DC

In any triangle ABC, find the ratio

$$\frac{sinA + sinB + sinC}{cotA + cotB + cotC}$$

function of the altitudes in the triangle.

Solution

In any triangle ABC, the next two relationships are true:

$$a^2 = b^2 + c^2 - 2abccosA$$

and

$$S = \frac{1}{2}bcsinA$$

From these, we obtain

$$cotA = \frac{b^2 + c^2 - a^2}{4S}.$$

Consequently

$$\frac{sinA + sinB + sinC}{cotA + cotB + cotC} = \frac{\frac{2S}{bc} + \frac{2S}{ac} + \frac{2S}{ab}}{\frac{b^2+c^2-a^2}{4S} + \frac{a^2+c^2-b^2}{4S} + \frac{a^2+b^2-c^2}{4S}} = \frac{8S^2(a+b+c)}{abc(a^2+b^2+c^2)}.$$

We replace S with $s(s-a)(s-b)(s-c)$ or S^2 with $\frac{1}{16}(a+b+c)(b+c-a)(a+c-b))a+b-c)$ obtaining

$$\frac{sinA + sinB + sinC}{cotA + cotB + cotC} = \frac{(a+b+c)^2(b+c-a)(a+c-b)(a+b-c)}{2abc(a^2+b^2+c^2)}$$

We know that

$$S = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$$

and the expression becomes

$$\frac{sinA+sinB+sinC}{cotA+cotB+cotC}=\frac{(\frac{2S}{h_a}+\frac{2S}{h_b}+\frac{2S}{h_c})^2(\frac{2S}{h_b}+\frac{2S}{h_c}-\frac{2S}{h_a})(\frac{2S}{h_a}+\frac{2S}{h_c}-\frac{2S}{h_b})(\frac{2S}{h_a}+\frac{2S}{h_b}-\frac{2S}{h_c})}{2\frac{2S}{h_a}\frac{2S}{h_b}\frac{2S}{h_c}((\frac{2S}{h_b})^2+(\frac{2S}{h_b})^2+(\frac{2S}{h_b})^2)}$$

$$\frac{sinA+sinB+sinC}{cotA+cotB+cotC}=(h_ah_b+h_bh_c+h_ah_b)^2\frac{(h_ah_c+h_ah_b-h_bh_c)(h_bh_c+h_ah_b-h_ah_c)(h_bh_c+h_ah_c-h_ah_b)}{2h_a^2h_b^2h_c^2(h_a^2h_b^2+h_b^2h_c^2+h_a^2h_c^2)}$$

Problem 28A. Proposed by Max Jiang

The number 2021 is written on a blackboard. Alice and Bob play a game where they take turns erasing the current number on the blackboard, n, and write either $n-1$ or $n-336$. They cannot write negative integers and the player who writes 0 is the winner. Given that Alice goes first, show that Alice has a winning strategy.

Solution

Call n a winning position if Alice has a winning strategy if it is her turn when n is written on the blackboard and a losing position otherwise. We wish to prove that 2021 is a winning position.

For $1 \leq n < 336$, note that since each player can only replace n with $n-1$ on their turn, the parity of the number on the blackboard must change every turn.
If n is odd and Alice goes first, she will always write even numbers on her turns and similarly Bob will always write odd numbers. Thus, only Alice can write 0 so n is a winning position for odd $1 \leq n < 336$. Similarly, n is a losing position for even $1 < n < 336$. Additionally, 336 is clearly a winning position since Alice can write $336-336=0$ on her first turn.

We now prove by strong induction that n is a winning position if and only if $n \equiv k \pmod{337}$ for some $k \in S = \{1, 3, 5, \dots, 333, 335, 336\}$. In other words, k is either 366 or an odd natural less than 337. Note that we have already shown this for $1 \leq n \leq 336$.

Now, let us assume our hypothesis holds for all naturals less than some $n > 336$. If $n \equiv k \pmod{337}$ for an odd $1 \leq k < 337$, then $n-1 \equiv k-1 \pmod{337}$ where $0 \leq k-1 < 336$ is even. Thus, we see that $k-1 \notin S$, so $n-1$ is a losing position by our inductive hypothesis. Since Alice can write $n-1$ to leave a losing position for Bob, her position must be winning in this case.

If $n \equiv 336 \pmod{337}$, Alice can write $n-336 \equiv 0 \pmod{337}$. Since $n-336$ is a losing position, Alice's position is winning in this case as well. Together, these cases show that n is a winning position if $n \equiv k \pmod{337}$ for $k \in S$.

Now, consider when $n \equiv k \pmod{337}$ for $k \notin S$, in other words, we have an even $0 \le k < 336$. If $k = 0$, then $n - 1 \equiv 336 \pmod{337}$ and $n - 336 \equiv 1 \pmod{337}$. We see that both are winning positions, so n must be a losing position in this case.
For $k > 0$, note that $n - 1 \equiv k - 1 \pmod{337}$ and $n - 336 \equiv k + 1 \pmod{337}$. We have $0 < k - 1 < k + 1 < 337$ are both odd and thus belong to S. Thus, they are both winning positions, meaning that n must be a losing position in this case as well. Together, these cases show that n is a losing position if $n \equiv k \pmod{337}$ for $k \notin S$.

Thus, we have shown that n is a winning position if and only if $n \equiv k \pmod{337}$ for $k \in S$, which completes our inductive step and thus our proof. Finally, note that $2021 \equiv 336 \pmod{337}$, so 2021 is in fact a winning position.

Problem 29A. Proposed by Vedaant Srivastava

At the annual Obscure Meeting for Journalists, there are to be 2021 journalists seated around a circular table. Each journalist is assigned a number from 1 to 2021 depending on their political polarization. For each pair of journalists sitting next to one another, their *disagreement* is defined as the absolute difference between their assigned numbers. The *disagreement index* for a given table arrangement is the sum of the disagreement of all adjacent journalists. Given that the journalists sit around the table uniformly at random, what is the expected value of the disagreement index?

Solution

Note that finding the disagreement index for each table configuration is very tedious. We will thus make use of linearity of expectation.

Number the chairs from 1 to 2021.
Let X be the disagreement index for a given table configuration.
Let X_i be the disagreement of journalists sitting in chairs i and $i + 1$, where

indices are taken modulo 2021. Thus we have that

$$X = \sum_{i=1}^{2021} X_i$$

$$\mathbb{E}[X] = \sum_{i=1}^{2021} \mathbb{E}[X_i]$$

$$\mathbb{E}[X] = \sum_{i=1}^{2021} \frac{1}{\binom{2021}{2}} \cdot \sum_{j=1}^{2020} j(2021 - j)$$

$$\mathbb{E}[X] = \sum_{i=1}^{2021} \frac{1}{\binom{2021}{2}} \cdot \sum_{j=1}^{2020} 2021j - j^2$$

$$\mathbb{E}[X] = \sum_{i=1}^{2021} \frac{1}{\binom{2021}{2}} \cdot \left(2021\frac{2020 \cdot 2021}{2} - \frac{2020 \cdot 2021 \cdot 4041}{6}\right)$$

$$\mathbb{E}[X] = \sum_{i=1}^{2021} \frac{2022}{3}$$

$$\mathbb{E}[X] = \boxed{\frac{2021 \cdot 2022}{3}}$$

Where the third line follows as for a given disagreement j there are $2021 - j$ possible unordered pairs of journalists, with $\binom{2021}{2}$ total possible unordered pairs of journalists.

Problem 30A. Proposed by Alexander Monteith-Pistor

Let $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ be four line segments of length 10. For each pair $1 \leq i < j \leq 4$, the line segments A_iB_i and A_jB_j intersect at point P_{ij}. Starting at A_1 and travelling along the four line segments, find the least upper bound for the distance one has to travel to pass through all 6 points of intersection (P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}).

Problem 31A. Proposed by Andy Kim

Find the volume of a rectangular prism that has faces with diagonals of length 8, 10, and 12.

Solution

Proof. Let the three side lengths of the rectangular prism be x, y, and z. Then, we have

$$x^2 + y^2 = 8^2$$
$$x^2 + z^2 = 10^2$$
$$y^2 + z^2 = 12^2$$

Then, we have

$$x^2 = \frac{8^2 + 10^2 - 12^2}{2} = 10$$
$$y^2 = \frac{8^2 + 12^2 - 10^2}{2} = 54$$
$$z^2 = \frac{10^2 + 12^2 - 8^2}{2} = 90$$

This gives

$$x = \sqrt{10}, y = 3\sqrt{6}, z = 3\sqrt{10}$$

and thus the volume is

$$xyz = (\sqrt{10})(3\sqrt{6})(3\sqrt{10}) = 90\sqrt{6}$$

□

Problem 32A. Proposed by Nicholas Sullivan

The Riemann zeta function is defined $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$, and has lots of interesting properties. For example, the sum of inverse squares is equal to $\zeta(2) = \frac{\pi^2}{6}$. If we let $a_n = \zeta(n) - 1$, then show that:

$$\sum_{n=1}^{\infty} a_{2n} = 3 \sum_{n=1}^{\infty} a_{2n+1}.$$

Solution

The left-hand sum, S_1, can expressed more explicitly as:

$$S_1 = \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k^{2n}}.$$

Since these infinite series are nice and convergent, we can switch the order of summation to get:

$$S_1 = \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{2n}}.$$

The inner series is now in the form of an infinite geometric series, and since the ratio $|r| = |\frac{1}{k^2}| < 1$, this series converges and can be replaced with:

$$S_1 = \sum_{k=2}^{\infty} \frac{1}{k^2 - 1}.$$

The term $\frac{1}{k^2-1}$ can be split into two terms $\frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$, which gives:

$$S_1 = \sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

This sum telescopes, leaving only the first two terms:

$$S_1 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

Similarly, with the right-hand sum, S_2, we have:

$$S_2 = \sum_{n=1}^{\infty}\sum_{k=2}^{\infty} \frac{1}{k^{2n+1}}.$$

If we add S_1, then this can be rewritten (since the even power terms are now included):

$$S_1 + S_2 = \sum_{n=2}^{\infty}\sum_{k=2}^{\infty} \frac{1}{k^n}.$$

Again, we can switch the order of summation and use the sum of geometric series to get:

$$\begin{aligned} S_1 + S_2 &= \sum_{k=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{k^n} \\ &= \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) \end{aligned}$$

This sum telescopes, leaving only the first term:

$$S_1 + S_2 = 1.$$

Thus, $S_2 = 1 - \frac{3}{4} = \frac{1}{4}$, and so $S_1 = 3S_2$.

Problem 33A. Proposed by DC

On circle with diameter AB, take two points C and D and build the intersection between AC and BD denoted as P. Prove that

$$BD^2 - BC^2 = AC \times PC + BD \times DP$$

Problem 34A. Proposed by DC

In triangle ABC with angle $A = 45°$, build D, the projection of B on AC and E, the projection of C on AB. Find the ratio between the area of triangle ADE and area of triangle ABC.

Solution

The quadrilateral EDCB is cyclic ($m(\sphericalangle CDB) = m(\sphericalangle BEC) = 90°$. Thus, $m(\sphericalangle AED) = m(\sphericalangle ACB)$ and $m(\sphericalangle ADE) = m(\sphericalangle ABC)$. From the last two relationships we conclude:
$\Delta ADE \sim \Delta ABC$ (AA) and

$$\frac{Area\Delta ADE}{Area\Delta ABC} = \left(\frac{AE}{AC}\right)^2$$

Triangle AEC is right angled and isosceles $[m(\sphericalangle CAE) = m(\sphericalangle ACE) = 45°]$. Thus, $2AE^2 = AC^2$. Consequently,

$$\frac{Area\Delta ADE}{Area\Delta ABC} = \left(\frac{AE}{AC}\right)^2 = \frac{1}{2}$$

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 29B. Proposed by David Andrei Anghel

There are n fixed lines $(l_1, l_2, \ldots, l_n)$ in a given plane, with n an odd natural number greater then 1. P_0 is a moving point on the same plane. For each $i \in \{1, 2, \ldots, 2n\}$ define P_i as the symmetric of P_{i-1} with respect to line l_i (the indices are considered *modulo* n). Show that vector P_0P_{2n} does not depend on the position of P_0.

Solution

The solution uses the complex plane, with the complex number t assigned to point T. The requirement is equivalent to demonstrating that $p_{2n} - p_0$ does not depend on p_0. For any complex number u we will denote $f(u)$ its conjugate and $f^k(u) = f(f(\ldots f(u)\ldots)))$ where f is applied k times. The symmetry with respect to the line is the transformation $y \longrightarrow xf(y) + c$ where x and c do not depend on y and $|x| = 1$. Consequently, $p_{2n} = f^{2n}(p_0) * \prod_1^{2n} f^i(x_i) +$ *constant*, where x_i is defined for line l_i (the indices are considered *modulo* n).

Furthermore, f^2 is the identity function and $f^{2n}(p_0) = p_0$. In addition, for each $i \in \{1, 2, \dots, 2n\}$:

$$f^i(x_i) * f^{i+n}(x_{i+n}) = x_i f(x_i) = |x_i|^2 = 1$$

because n is odd. Consequently, $p_{2n} = p_0 + constant$.

Problem 30B. Proposed by Max Jiang

Given a string of a's and b's, we can replace a substring using the following operations:

$$\begin{aligned}
\underbrace{aaa\cdots a}_{k}\underbrace{b\cdots bbb}_{k} &\implies \underbrace{abab\cdots ab}_{2k},\\
\underbrace{abab\cdots ab}_{2k} &\implies \underbrace{aaa\cdots a}_{k}\underbrace{b\cdots bbb}_{k},\\
\underbrace{bbb\cdots b}_{k}\underbrace{a\cdots aaa}_{k} &\implies \underbrace{baba\cdots ba}_{2k},\\
\underbrace{baba\cdots ba}_{2k} &\implies \underbrace{bbb\cdots b}_{k}\underbrace{a\cdots aaa}_{k},
\end{aligned}$$

where $k \in \mathbb{N}$. For example, the following operations are valid:

$$\begin{aligned}
\underbrace{aabb}\,ababbbaababa &\implies abab\,\underbrace{abab}\,bbaababa\\
&\implies ababaabb\,\underbrace{bbaa}\,baba\\
&\implies ababaabbbaba\,\underbrace{baba}\\
&\implies ababaabbbababbaa.
\end{aligned}$$

Given the string

$$\underbrace{aaa\cdots a}_{n}\underbrace{bbb\cdots b}_{n},$$

find all strings that are attainable by repeatedly applying the four operations.

Solution

We claim that the answer is all strings that begin with a, end with b, and have an equal number of a's and b's. Note that all four operations preserve the first and last letters of string, as well as the number of a's and b's, so these conditions are necessary.

Note that since the inverse of each operation is also a valid operation, it is equivalent to find all strings such that a sequence of operations will transform it into our initial string

$$\underbrace{aaa\cdots a}_{n}\underbrace{bbb\cdots b}_{n}.$$

Let us proceed using strong induction. For our case $n = 1$, the only string that satisfies our conditions is ab, which is clearly attainable from itself.

Now, let us assume that any string starting with a, ending in b, and containing k a's and k b's, for all naturals $k < n$ can be transformed into

$$\underbrace{aaa\cdots a}_{k}\underbrace{bbb\cdots b}_{k}$$

using the four operations.
Now, consider a string starting with a, ending in b, and containing n a's and n b's. We have four cases.

Case 1: The 2nd character of the string is a and the $2n-1$th is b.
We see that the substring containing every character except the first and last is a string starting with a, ending in b, and containing $n-1$ a's and $n-1$ b's, so we have

$$a\underbrace{a\cdots b}_{2(n-1)}b \implies a\underbrace{aaa\cdots a}_{n-1}\underbrace{bbb\cdots b}_{n-1}b,$$

which is clearly equivalent to $\underbrace{aaa\cdots a}_{n}\underbrace{bbb\cdots b}_{n}$, as desired.

Case 2: The 2nd character is b and the $2n-1$th is a.
Since each operation has a "mirrored" counterpart, we see there is a sequence of operations that transforms

$$a\underbrace{b\cdots a}_{2n-2}b \implies a\underbrace{bbb\cdots b}_{n-1}\underbrace{aaa\cdots a}_{n-1}b.$$

Then, we have

$$\begin{aligned} a\underbrace{bbb\cdots b}_{n-1}\underbrace{aaa\cdots a}_{n-1}b &\implies a\underbrace{baba\cdots ba}_{2(n-1)}b \\ &\cong \underbrace{ababa\cdots bab}_{2n} \\ &\implies \underbrace{aaa\cdots a}_{n}\underbrace{bbb\cdots b}_{n}. \end{aligned}$$

Case 3: The 2nd and $2n-1$th characters are a.
Since the last two characters are ab, the prefix of length $2n-2$ must have an equal number of a's and b's. Let us consider the shortest prefix that has an equal number of a's and b's. Clearly, any prefix of the string begins with a and we notice that this prefix it must end in b, otherwise it would not be the shortest with an equal number of a's and b's. Additionally, it has length $2k \le 2n-2$, so there must be a sequence of operations that transforms our string as such:

$$\underbrace{aa\cdots b}_{2k}\ \underbrace{\cdots}_{2n-2k-2}\ ab \implies \underbrace{aaa\cdots a}_{k}\underbrace{bbb\cdots b}_{k}\ \underbrace{\cdots}_{2n-2k-2}\ ab,$$

and then we have

$$\underbrace{aaa\cdots a}_{k}\underbrace{bbb\cdots b}_{k}\underbrace{\cdots}_{2n-2k-2}\ ab \implies \underbrace{abab\cdots ab}_{2k}\underbrace{\cdots}_{2n-2k-2}\ ab$$

Now, note that the substring containing every character except the first and last starts in b, ends in a, and contains an equal number of a's and b's. Thus, there is a sequence of operations that transforms our string as such:

$$a\underbrace{b\cdots a}_{2(n-1)}b \implies a\underbrace{bbb\cdots b}_{n-1}\underbrace{aaa\cdots a}_{n-1}b.$$

Then, we have

$$\begin{aligned} a\underbrace{bbb\cdots b}_{n-1}\underbrace{aaa\cdots a}_{n-1}b &\implies a\underbrace{baba\cdots ba}_{2(n-1)}b \\ &\cong \underbrace{abab\cdots ab}_{2n} \\ &\implies \underbrace{aaa\cdots a}_{n}\underbrace{bbb\cdots b}_{n}, \end{aligned}$$

as desired.

Case 4: The 2nd and $2n-1$th characters are b.
Note that this case is equivalent to case 3 after replacing each a with b and vice versa, and then "mirroring" the resulting string. Since applying this transformation to each of the operations results in another valid operation, we see that a similar method shows that the string can be transformed into our desired string in this case as well.
In particular, we have for some $2 \le k \le n-1$,

$$\begin{aligned} ab\underbrace{\cdots}_{2n-2k-2}\underbrace{a\cdots bb}_{2k} &\implies ab\underbrace{\cdots}_{2n-2k-2}\underbrace{aaa\cdots a}_{k}\underbrace{bbb\cdots b}_{k} \\ &\implies ab\underbrace{\cdots}_{2n-2k-2}\underbrace{abab\cdots ab}_{2k} \\ &\cong a\underbrace{b\cdots a}_{2(n-1)}b \\ &\implies a\underbrace{bb\cdots b}_{n-1}\underbrace{aa\cdots a}_{n-1}b \\ &\implies a\underbrace{bab\cdots a}_{2(n-1)}b \\ &\cong \underbrace{abab\cdots ab}_{2n} \\ &\implies \underbrace{aa\cdots a}_{n}\underbrace{bb\cdots b}_{n}. \end{aligned}$$

This completes our proof.

Problem 31B. Proposed by Vedaant Srivastava

The Organization of Mathematicians in Jeopardy is hosting a new years ping pong tournament! There are 2021 mathematicians taking part in the tournament, where each competitor plays against each other exactly once. In each game, there is one winner and one loser. Given competitors x, y, z, we say that a *triple-upset* has happened if x wins against y, y wins against z, and z wins against x. What is the maximum possible number of triple-upsets in the tournament?

Solution

Let us represent the tournament as a complete directed graph on vertices v_i $1 \le i \le 2021$ where we draw the edge $v_i \to v_j$ if competitor i wins against j and $v_j \to v_i$ if j wins against i.

Thus the problem reduces to finding the maximum number of directed triangles.

Let T be the number of directed triangles and T' be the number of non-directed triangles. Thus we have that

$$T + T' = \binom{2021}{3} \tag{1}$$

Thus, to find the maximum value of T, it suffices to find the minimum value of T'.

Let $\deg^-(v_i)$ denote the indegree of the vertex v_i. Note that the average value of $\deg^-(v_i)$ is

$$\overline{\deg^-(v_i)} = \frac{1}{2021}\sum_{i=1}^{2021} \deg^-(v_i) = \frac{1}{2021}\binom{2021}{2} = 1010$$

Furthermore, observe that each non-directed triangle with vertices v_x, v_y, v_z has some unique vertex v_x with edges $(v_y \to v_x)$, $(v_z \to v_x)$. This implies that

$$T' = \sum_{i=1}^{2021} \binom{\deg^-(v_i)}{2} \ge 2021 \cdot \binom{\overline{\deg^-(v_i)}}{2} = 2021 \cdot \binom{1010}{2} \tag{2}$$

by convexity.

Thus, (1) and (2) imply that

$$T = \binom{2021}{3} - T' \le \boxed{\binom{2021}{3} - 2021 \cdot \binom{1010}{2}}$$

Construction:
For each edge (v_i, v_j) with $i < j$, construct the directed edge $(v_i \to v_j)$ if $j - i$

is odd, otherwise construct $(v_j \to v_i)$ if $j-i$ is even.

This configuration yields the desired result.

Problem 32B. Proposed by Alexander Monteith-Pistor

An arithmetic sequence is called maximal in a set S if each of its elements are distinct and in S, and it is not contained in a larger such arithmetic sequence. For example, $1,2,3,4$ is maximal in $\{1,2,3,4,6\}$ while $1,2,3$ is not. Similarly, a geometric sequence is called maximal in a set S if each of its elements are distinct and in S, and it is not contained in a larger such geometric sequence.

Let $n \geq 3$. Let A be the smallest integer such that every set of n positive integers has at most A maximal arithmetic sequences. Let G be the smallest integer such that every set of n positive integers has at most G maximal geometric sequences. Prove that $A = G$.

Solution

Let S be a set of n positive integers with A maximal arithmetic sequences. Let the elements of S be $x_1 < x_2 < ... < x_n$. Consider the set

$$S' = \{2^{x_1}, 2^{x_2}, ..., 2^{x_n}\}$$

For all positive integers $a_1, ..., a_k$ (for some $k \geq 3$), $a_1, a_2, ..., a_k$ is an arithmetic sequence if and only if $2^{a_1}, 2^{a_2}, ..., 2^{a_k}$ is a geometric sequence. Thus, every maximal arithmetic sequence in S is a maximal geometric sequence in S'. Since S' is a set of n distinct positive integers, $G \geq A$.

Alternatively, let S be a set of n positive integers with G maximal geometric sequences. Let the elements of S be $x_1 < x_2 < ... < x_n$. Consider the set of real numbers

$$S' = \{\log_2 x_1, \log_2 x_2, ..., \log_2 x_n\}$$

For all real numbers $g_1, ..., g_k$ (for some $k \geq 3$), $g_1, g_2, ..., g_k$ is a geometric sequence if and only if $\log_2 g_1, \log_2 g_2, ..., \log_2 g_k$ is an arithmetic sequence.

We want to convert S' into a set of positive integers and preserve the maximal arithmetic sequences. First, we translate S' (add $1 - \log_2 x_1$ to each element) to obtain the set $\{1, r_2, ..., r_n\}$ where $1 < r_2 < ... < r_n$.

Next, let $d_1 = r_2 - 1$, and $d_i = r_{i+1} - r_i$ for $2 \leq i \leq n-1$. Let C be a positive integer such that Cd_i is within $\frac{1}{2n}$ of an integer for all $1 \leq i \leq n-1$ (see *). We can impose the condition that each Cd_i rounds to a different integer (see *). Finally, consider the set obtained by adjusting the differences:

$$\{1, 1 + C(r_2 - 1), 1 + C(r_3 - 1), ..., 1 + C(r_n - 1)\}$$

If we round every element of the set to the nearest integer, arithmetic sequences are preserved and no new ones are formed (details are left to the reader). Hence, this new set of distinct positive integers implies $A \geq G$ which concludes the proof.

(*): consider the tuple $(cd_1, ..., cd_{n-1})$ for all positive integers c. We can represent the tuple by rounding each cd_i to the nearest rational number $\frac{m}{4n}$ for some integer m and letting t_i be the fractional part of $\frac{m}{4n}$ $(\frac{m \mod 4n}{4n})$. Doing so, we obtain a new tuple of rational numbers $(t_1, ..., t_{n-1})$ which is dependent on c. Notice there are only finitely many such rational tuples therefore we can find c_1, c_2 such that the resulting approximation is the same. We can let $C = |c_1 - c_2|$ and it follows that Cd_i is within $\frac{1}{2n}$ of an integer for each $1 \leq i \leq n-1$. Finally, we can preemptively multiply each d_i by $\frac{1}{D}$ where $D = \min(d_2 - d_1, d_3 - d_2, ..., d_{n-1} - d_{n-2})$ in order to ensure that each Cd_i rounds to a different integer.

Problem 33B. Proposed by Andy Kim

Let $p(x)$ be a polynomial with nonnegative integer coefficients. Prove or disprove the following statement: it is possible to determine every coefficient of $p(x)$ by evaluating it at 2 values.

Solution

Let $p(x)$ be a polynomial with nonnegative integer coefficients. Prove or disprove the following statement: it is possible to determine every coefficient of $p(x)$ by evaluating it at 2 values.

Proof. This statement is *true.*

Let the coefficients of $p(x)$ be a_i. First, we evaluate $p(1)$, and let $k = p(1)+1$. Here, since $p(1)$ is the sum of the coefficients and all coefficients are nonnegative, we have that $a_i \leq m - 1$ for all i.

Now, we evaluate $p(m)$. Then, noting that $a_j m^j$ is divisible by m^{i+1} for $j > i$, we have

$$p(m) \equiv \sum_{j=0}^{i} a_j m^j \pmod{m^{i+1}}$$

Also, for any i, we have

$$a_i m^i \leq (m-1)m^i = m^{i+1} - m^i$$

and therefore,

$$\sum_{j=0}^{i} a_j m^j \leq m^{i+1} - m^0 = m^{i+1} - 1$$

Noting that all coefficients are nonnegative, we have

$$0 \leq \sum_{j=0}^{i} a_j m^j < m^{i+1}$$

Then, letting r_i be the remainder when $p(m)$ is divided by m^{i+1}, we have

$$\sum_{j=0}^{i} a_j m^j = r_i$$

for all i. From this, we can calculate the coefficients as follows

$$a_0 = r_0$$
$$a_i = \frac{1}{m^i}(r_i - r_{i-1})$$

Note: another way to do this would be to pick k with $10^k > p(1)$ and look at $p(10^k)$. The coefficients of p will be in the decimal expansion of $p(10^k)$, separated by some number of zeros.

□

Problem 34B. Proposed by Nicholas Sullivan

Let A, B, C and D be arbitrary points on the plane as seen in the figure below. Let A, B and C lie on circle c_1, let A, B and D lie on circle c_2 and let A, C and D lie on circle c_3. Let E be the intersection of the tangents at B and C to circle c_1, let F be the intersection of the tangents at B and D to circle c_2, and let G be the intersection of the tangents at C and D to circle c_3, as in the figure. Show that

$$\mathrm{m}(\angle EBF) + \mathrm{m}(\angle FDG) + \mathrm{m}(\angle GCE) = 180^\circ.$$

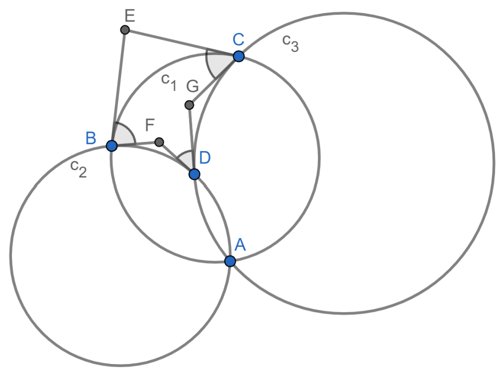
Solution

We decided to publish the solution proposed by Andrew Dong.

Using the given diagram, note that :

$$\mathrm{m}(\angle BFD) = 180^\circ - 2\mathrm{m}(\angle FBD) = 180^\circ - 2\mathrm{m}(\angle BAD)$$

$$\mathrm{m}(\angle CGD) = 180^\circ - 2\mathrm{m}(\angle GCD) = 180^\circ - 2\mathrm{m}(\angle CAD)$$



$$\text{m}(\angle BEC) = 180^\circ - 2\text{m}(\angle CBE) = 180^\circ - 2\text{m}(\angle BAC)$$

Let $x = \text{m}(\angle EBF) + \text{m}(\angle FDG) + \text{m}(\angle GCE)$. Using appropriate reflex angles, we get that the sum of the interior angles of hexagon BFDGCE is

$$(2\text{m}(\angle BAD)+180^\circ)+(2\text{m}(\angle CAD)+180^\circ)+(180^\circ-2\text{m}(\angle BAC)+x = 540^\circ+x.$$

On the other hand, we know this value must be equal to 720° as it is the sum of the interior angles of a non-self-intersecting hexagon. Hence, $x = 180^\circ$.

Problem 35B. Proposed by DC

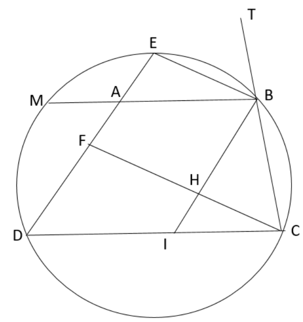
Consider trapezoid $ABCD$ with AD=6 cm and BC=5 cmc, and parallel sides $AB < CD$. If E is the intersection of AD with the circle on BCD and F is the intersection of AD with the parallel from C to BE, find the length of EF.

Solution

We have $m(\sphericalangle FCB) = m(\sphericalangle EBT)$ where BT is the extension of BC and arc BC=arc DM, arcs between parallel lines.

$m(\sphericalangle BAE) = m(\sphericalangle EBT)$ $(m(\sphericalangle EBT) = \frac{1}{2}(arcBC + arcEB) = \frac{1}{2}(arcDM + arcEB) = m(\sphericalangle BAE)$. If $m(\sphericalangle BAE) = m(\sphericalangle FCB)$ then ABCF cyclic.
Build BI the parallel from B to AD with I on DC and obtain H the intersection of BI and CF. $m(\sphericalangle BAE) = m(\sphericalangle BIC)$ (parallel sides $EF \parallel BI$ and $AB \parallel CD$). Thus, $m(\sphericalangle FCB) = m(\sphericalangle BIC)$

$\Delta BCI \sim \Delta BHC$ (AA) $[m(\sphericalangle BIC) = m(\sphericalangle HCB)$ and $\sphericalangle CBI$ is common$]$. From the proportionality of the sides, $BC/BI = BH/BC$; knowing that $BI = AD$ and $BH = EF$, we obtain $EF = BC^2/AD$. Consequently, $EF = 25/6$ cm.



Problem 36B. Proposed by DC

Consider triangle ABC with $AB = 5$ cm and $BC = 6$ cm. Build AH, the altitude and take point P on AH such that the circle with the origin at P and radius AP will intersect side AB at M and side AC at N. Build a second circle on M and N that intersects side AB at R and side AC at S. If P and the second circle are built such that $AR = 4$ cm, find RS.

Solution

The quadrilateral RMNS is cyclic and we obtain $m(\sphericalangle AMN) = m(\sphericalangle ASR)$. Buid AT the tangent in A to the circle AMN. $m(\sphericalangle AMN) = m(\sphericalangle NAT)$. Consequently, $m(\sphericalangle ASR) = m(\sphericalangle NAT)$

$AT \parallel BC$ and $m(\sphericalangle ACB) = m(\sphericalangle NAT)$. Thus, $m(\sphericalangle ASR) = m(\sphericalangle ACB)$ and $RS \parallel BC$. From the last relationship, $\Delta ARS \sim \Delta ABC$ (AA). From the proportionality of the sides, $AR/AB = RS/BC$ and $RS = BC \times (AR/AB) = 6 \times (4/5)$. Finally $RS = 24/5$.