Number 7 - July 2021 Problems

August 18, 2021

Problems

Problem 35A. Proposed by DC

In trapezoid ABCD, the bases are AB=7 cm and CD=3 cm. The circle with the origin at A and radius AD intersects diagonal AC at M and N. Calculate the value of the product $CM \times CN$.

Problem 42A. Proposed by Vedaant Srivastava

Given positive reals a, b, c, prove that

$$\frac{a(a^3+1)}{2b+6c} + \frac{b(b^3+1)}{2c+6a} + \frac{c(c^3+1)}{2a+6b} \geq \frac{1}{8}(a^3+b^3+c^3+3)$$

Solution Problem 42A

From Cauchy-Schwartz, we have that

$$\left(\sum_{\text{cyc}} \frac{a^4}{2b+6c}\right)\left(\sum_{\text{cyc}} a^2(2b+6c)\right) \geq (a^3+b^3+c^3)^2$$

This implies that

$$\left(\sum_{\text{cyc}} \frac{a^4}{2b+6c}\right) \geq \frac{(a^3+b^3+c^3)^2}{\sum_{\text{cyc}} a^2(2b+6c)} \geq \frac{1}{8}(a^3+b^3+c^3) \tag{1}$$

Where the last inequality follows as $8(a^3+b^3+c^3) \geq \sum_{\text{cyc}} a^2(2b+6c)$ by standard AM-GM / Muirhead.

Again from Cauchy-Schwartz, we have that

$$\left(\sum_{\text{cyc}} \frac{a}{2b+6c}\right)\left(\sum_{\text{cyc}} a(2b+6c)\right) \geq (a+b+c)^2$$

This implies that

$$\left(\sum_{\text{cyc}} \frac{a}{2b+6c} \right) \geq \frac{(a+b+c)^2}{\sum_{\text{cyc}} a(2b+6c)} \geq \frac{3}{8} \tag{2}$$

Where the last inequality follows as $8(a+b+c)^2 \geq 3\sum_{\text{cyc}} a(2b+6c)$ by standard AM-GM / Muirhead.

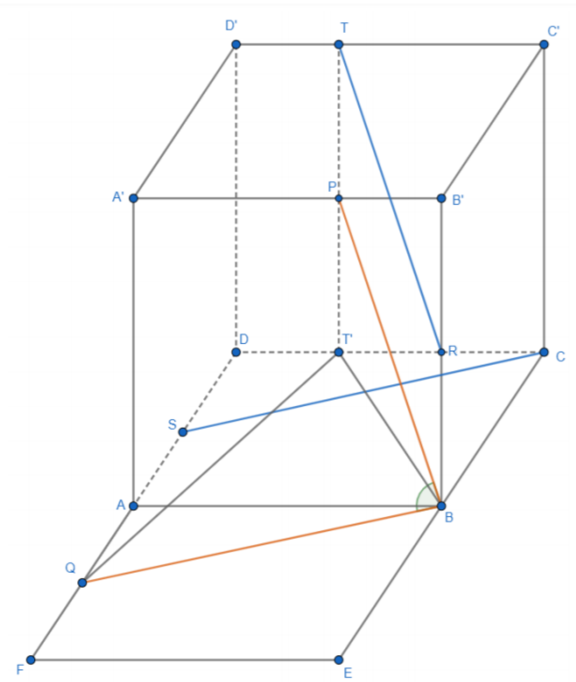
Adding (1) and (2) yields the desired result.

Problem 43A. Proposed by Alexandru Benescu

Let $ABCDA'B'C'D'$ be a cube, R be the midpoint of BB', and S be the midpoint of AD. Point T is on $C'D'$ such that $\frac{C'T}{D'T} = 2$. Find $\cos \measuredangle(SC, TR)$.

Solution Problem 43A

Let $AB = 6x$. Thus, $TC' = 4x$ and $TD' = 2x$. Let T' be a point on CD such that $TT' \perp CD$. Thus, $T'C = 4x$ and $T'D = 2x$. Let P be the midpoint of TT' and let Q be an extension of DA such that $QA = AS$ (see figure below).



Since $TP = RB = 3x$ and $TP \parallel CC' \parallel BR$, we know that $RBPT$ is a parallelogram. Similarly, since $BC = QS = 6x$ and $BC \parallel QS$, we see that

$BQSC$ is a parallelogram. Therefore, $TR \parallel PB$ and $SC \parallel QB$. Consequently, $\cos(\measuredangle(SC, TR)) = \cos(\measuredangle PBQ)$.

From Pythagorean Theorem in $\triangle BCT'$, we have that

$$(BT')^2 = (CT')^2 + (BC)^2 = 16x^2 + 36x^2 = 52x^2.$$

Using Pythagorean Theorem in $\triangle BT'P$, we get that

$$BP = \sqrt{(PT')^2 + (BT')^2} = \sqrt{9x^2 + 52x^2} = x\sqrt{61}.$$

Furthermore, $BQ = \sqrt{(AQ)^2 + (AB)^2} = \sqrt{9x^2 + 36x^2} = x\sqrt{45}$.

Using Pythagorean Theorem in $\triangle QDT'$, we get $(QT')^2 = (QD)^2 + (DT')^2 = 81x^2 + 4x^2 = 85x^2$. Substituting this into Pythagorean Theorem for $\triangle PT'Q$, we get

$$PQ^2 = (PT')^2 + (QT')^2 = 9x^2 + 85x^2 = 94x^2.$$

From the Law of Cosines in $\triangle BPQ$, we have that

$$PQ^2 = PB^2 + QB^2 - 2PB \cdot QB \cdot \cos(\measuredangle(PBQ)).$$

Thus,

$$\begin{aligned}
\cos(\measuredangle(PBQ)) &= \frac{PB^2 + QB^2 - PQ^2}{2PB \cdot QB} \\
&= \frac{61x^2 + 45x^2 - 94x^2}{2 \cdot x\sqrt{61} \cdot x\sqrt{45}} \\
&= \frac{12}{6\sqrt{305}} \\
&= \frac{2}{\sqrt{305}} \\
&= \frac{2\sqrt{305}}{305}.
\end{aligned}$$

Problem 48A. Proposed by Vedaant Srivastava

Let G be the set of all lattice points (x, y) on the Cartesian plane where $0 \leq x, y \leq 2021$. Suppose that there are 2022 roadblocks positioned at points in G such that no two roadblocks have the same x or y coordinate. Anna starts at the point $(0, 0)$, attempting to reach the point $(2021, 2021)$ through a sequence of moves. In each move, she moves one unit up, down, left, or right, such that she always remains in G. Given that Anna cannot visit a lattice point which is occupied by a roadblock, determine all configurations of the roadblocks in which Anna is unable to reach her destination.

Solution Problem 48A

We call a *diagonal block* a set of roadblocks positioned at $(0, j), (1, j-1), \ldots, (j, 0)$ or $(2021, j), (2020, j+1), \ldots (j, 2021)$ for some $j \in \mathbb{Z} \mid 0 \leq j \leq 2021$.
Then, the following claim finishes the problem:

Claim: Anna is unable to reach her destination if and only if a diagonal block exists.

Proof. It can be easily verified that if a diagonal block exists, then Anna is unable to reach her destination.

We now prove that this is a necessary condition. Suppose that Anna is unable to reach her destination. Let S be the fixed set of points which Anna is able to reach. Consider the set of points on the boundary of S. Clearly, any two adjacent points on the boundary of S are at most 1 unit apart in the x-direction and y-direction. Otherwise, more points can be added to S, contradiction.

Furthermore, we have that at least one of the points on the boundary of S lies on the border of G, otherwise, S can be extended along the border of G, contradiction.

This implies that some point (p, q) on the boundary of G must be a roadblock. Then, as no two roadblocks have the same x or y coordinate and adjacent roadblocks must be at most one unit in each direction apart, the next roadblock must have the coordinates $(p + d_x, q + d_y)$, where $d_x, d_y = \pm 1$. Similarly, the next roadblocks after that must have the coordinates $(p + 2d_x, q + 2d_y), (p + 3d_x, q + 3d_y), \ldots$ and so on. This forms a diagonal block, as desired. □

Problem 50A. Proposed by Nicholas Sullivan

Let O be the intersection point of the two diagonals of non-degenerate quadrilateral $ABCD$. Next, let E, F, G and H be the midpoints of AB, BC, CD and DA respectively. If EG and FH intersect at O, show that $ABCD$ is a parallelogram.

Solution Problem 50A

(Solution 1) This can be shown using the vector representation. First, define $\vec{a} = OA$, $\vec{b} = OB$, $\vec{c} = OC$ and $\vec{d} = OD$. Thus, we can represent the midpoints as $OE = \frac{1}{2}(\vec{a} + \vec{b})$, $OF = \frac{1}{2}(\vec{b} + \vec{c})$, and so on.

Since O is the intersection of the diagonals of $ABCD$, we know that $\vec{a} = k_1\vec{c}$ and $\vec{b} = k_2\vec{d}$ for some scalars k_1, k_2. Similarly, since O is the intersection of EG

and FH, we can write that:

$$\frac{1}{2}(\vec{a}+\vec{b}) = \frac{k_3}{2}(\vec{c}+\vec{d})$$
$$\frac{1}{2}(\vec{b}+\vec{c}) = \frac{k_4}{2}(\vec{d}+\vec{a}),$$

for some scalars k_3 and k_4. Plugging in the expressions for $\vec{a}$ and $\vec{b}$, we have:

$$k_1\vec{c}+k_2\vec{d} = k_3(\vec{c}+\vec{d})$$
$$k_2\vec{d}+\vec{c} = k_4(\vec{d}+k_1\vec{c}).$$

If $\vec{c}$ and $\vec{d}$ are not linearly independent, then they can be written as scalar multiples of each other, and the vertices of the quadrilateral become collinear. Since $ABCD$ is non-degenerate, then $\vec{c}$ and $\vec{d}$ must be linearly independent.

Thus, the above equations can only be satisfied if:

$$k_1 = k_3$$
$$k_2 = k_3$$
$$k_2 = k_4$$
$$1 = k_4 k_1.$$

Thus, $k_1 = k_2 = k_3 = k_4 = \pm 1$. If we take the positive sign, then $\vec{a} = \vec{c}$ and $\vec{b} = \vec{d}$, causing $ABCD$ to be degenerate. Thus, we must take $\vec{a} = -\vec{c}$ and $\vec{b} = -\vec{d}$.

As a result, $\vec{a}-\vec{b} = \vec{d}-\vec{c}$ and $\vec{a}-\vec{d} = \vec{b}-\vec{c}$, so $AB \parallel DC$ and $AD \parallel BC$. Thus, $ABCD$ is a parallelogram.

(Solution 2) First, we note that O is the centroid of $ABCD$, with EG and FH both dividing the quadrilateral into two equal areas. That is, $S(ABFH) = S(DCFH)$ and $S(ADGE) = S(BCGE)$.

Next, consider triangle ABO. Since OE is a median, it divides the triangle into two equal areas, $S(AEO) = S(BEO)$. Let this be labelled s_1. Similarly, let $s_2 = S(BFO) = S(CFO)$, $s_3 = S(CGO) = S(DGO)$ and $s_4 = S(DHO) = S(AHO)$.

As a result, we know that:

$$S(ABFH) = S(DCFH)$$
$$2s_1 + s_2 + s_4 = 2s_3 + s_2 + s_4$$
$$S(ADGE) = S(BCGE)$$
$$2s_4 + s_1 + s_3 = 2s_2 + s_1 + s_3.$$

Thus, $s_1 = s_3$ and $s_2 = s_4$.

Since O is the centroid, then it is also the midpoint of EG and FH, implying that $OE = OG$ and $OF = OH$.

Next, consider triangles AEO and CGO. Since $s_1 = s_3$, they have equal areas, and by opposite angles, $\mathrm{m}(\measuredangle AOE) = \mathrm{m}(\measuredangle COG)$. Let this angle be denoted α.

Since their areas are equal, and $OE = OG$:

$$S(AEO) = S(CGO)$$
$$\frac{1}{2}OA \cdot OE \sin\alpha = \frac{1}{2}OC \cdot OG \sin\alpha$$
$$OA = OC.$$

Thus, O is the midpoint of AC, and by a similar argument, O is the midpoint of BD.

By SAS, triangles ABO and CDO are thus congruent ($OA = OC$, $\mathrm{m}(\measuredangle AOB) = \mathrm{m}(\measuredangle COD)$, $OB = OD$), implying that $AB = CD$. Similarly, ADO and CBO are congruent, implying that $AD = CB$.

Since $AB = CD$ and $AD = CB$, quadrilateral $ABCD$ is a parallelogram.

Problem 53A. Proposed by Nikola Milijevic

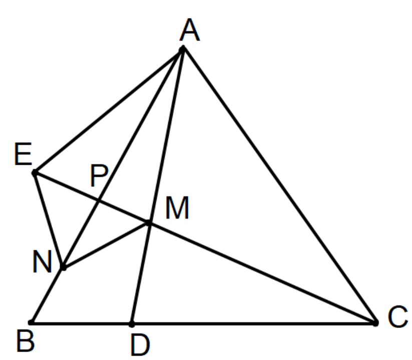
The positive integers $a_1, a_2, \ldots, a_n$ are not greater than 2021, with the property that $\mathrm{lcm}(a_i, a_j) > 2021$ for all $i, j, i \neq j$. Show that:

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2$$

Problem 54A. Proposed by Cosmina Ghitescu

Consider a point M in the interior of equilateral triangle ABC such that $m(\measuredangle MAC) = 45°$ and $m(\measuredangle MCB) = 30°$. Take $N \in (AB)$ such that $m(\measuredangle NMC) = 120°$. If $AM \cap CB = \{D\}$, find the value of the ratio $\frac{MN}{BD}$.

Solution Problem 54A



We will prove that $MN = BD$.

From exterior angle theorem in $\triangle AMC$, we have that $m(\measuredangle CMD) = m(\measuredangle MAC) + m(\measuredangle ACM) = 45° + 30° = 75°$. Via the sum of the angles in a triangle for

$\triangle ACD$, we get that $m(\measuredangle CDM) = 75^\circ$. Thus, $m(\measuredangle CMD) = m(\measuredangle CDM) = 75^\circ \Rightarrow \triangle MDC$ isosceles. $\Rightarrow MC = DC$.

We have that $MN + MC = MN + DC$. Let us prove that $MN + MC = BC$. Extend CM to E, where $EM = MN$. So, $MN + MC = EM + MC = EC$. Since $m(\measuredangle NME) = 180^\circ - m(\measuredangle NMC) = 180^\circ - 120^\circ = 60^\circ$, we have that $\triangle MNE$ is equilateral.

Consider $AB \cap CE = \{P\}$. Since $m(\measuredangle PAC) = 60^\circ$ and $m(\measuredangle ACP) = 30^\circ$, we know that $m(\measuredangle APC) = 90^\circ \Rightarrow NP \perp EM$. Because $\triangle MNE$ is equilateral, P must be the midpoint of EM.

Since $AP \perp EM$ and P is the midpoint of EM, we have that $\triangle AEM$ isosceles. Thus, AP is the angle bisector of $\measuredangle EAM$, meaning that $m(\measuredangle EAP) = m(\measuredangle PAM) = m(\measuredangle PAC) - m(\measuredangle MAC) = 15^\circ$. Therefore, $m(\measuredangle EAC) = 75^\circ$. Because $m(\measuredangle ECA) = 30^\circ$, we also have that $m(\measuredangle AEC) = 75^\circ$ via the sum of the angles in $\triangle AEC \Rightarrow EC = AC$.

We are given that $\triangle ABC$ is equilateral. Thus, $EC = AC = BC$.

Consequently, $MN + MC = BC = BD + DC$, meaning that $MN = BD$ and $\frac{MN}{BD} = 1$.

Problem 55A. Proposed by Eliza Andreea Radu

Consider the triangle ABC with $AB = 4$ cm, $BC = 6$ cm and $AC = 5$ cm. Take $M \in (AB)$ and $N \in (AC)$ such that $\cos(\measuredangle AMN) = \frac{3}{4}$. The feet of the perpendiculars drawn from B to MN, NC, and MC are P, Q, and R, respectively. What does P, Q, R form?

Problem 56A. Proposed by Cosmina Ghitescu

Let $ABCDA'B'C'D'$ be a cube with $M \in (BC)$ and $N \in (DD')$ such that $\frac{CM}{MB} = \frac{D'N}{ND} = k$. If $AC \cap DM = \{R\}$ and $CN \cap DC' = \{T\}$, find the value of k such that $RT||(ABC')$.

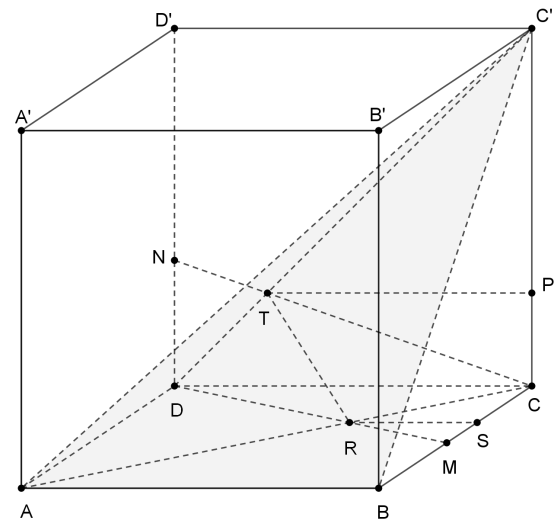
Solution Problem 56A

The segment RT must lie in a plane parallel to (ABC').

Let $S \in (BC)$ such that $RS \parallel AB$. Since $RT \parallel (ABC')$ and $RS \parallel AB$, we have that $(RST) \parallel (ABC')$.

Let $P \in (CC')$ such that $TP \parallel DC$. Since $RT \parallel (ABC')$ and $TP \parallel DC \parallel AB$, we have that $(PTR) \parallel (ABC')$.

Therefore, $(RTPS) \parallel (ABC')$, meaning that $PS \parallel BC'$.



By Thales in $\triangle BCC'$, we have that

$$\frac{C'P}{PC} = \frac{BS}{SC}. \tag{1}$$

Similarly, Thales in $\triangle CDC'$ yields

$$\frac{C'P}{PC} = \frac{C'T}{TD}.$$

Since $\triangle C'TC \sim \triangle DTN$ via AA similarity, we have that

$$\frac{C'T}{TD} = \frac{C'C}{ND}.$$

Thus, we get that $\dfrac{C'P}{PC} = \dfrac{C'C}{ND}$.

We are given that $\dfrac{D'N}{ND} = k$, which yields

$$\frac{D'N + ND}{ND} = \frac{DD'}{ND} = k + 1.$$

Because $DD' = CC'$, we have that $\dfrac{CC'}{ND} = k + 1$.

Therefore,

$$\frac{C'P}{PC} = k + 1. \tag{2}$$

By Thales in $\triangle ABC$, we have that

$$\frac{BS}{SC} = \frac{AR}{RC}.$$

We also have that $\triangle ADR \sim \triangle CMR$, giving us that

$$\frac{AR}{RC} = \frac{AD}{CM}.$$

Thus, we can conclude that $\dfrac{BS}{SC} = \dfrac{AD}{CM}$.

We are given that $\dfrac{CM}{MB} = k$, meaning that

$$\frac{CM}{CB} = \frac{k}{k+1}.$$

Taking the reciprocal and using the fact that $CB = AD$, we get

$$\frac{AD}{CM} = \frac{k+1}{k}.$$

Thus, we have that

$$\frac{BS}{SC} = \frac{k+1}{k}. \tag{3}$$

From (2): $\dfrac{C'P}{PC} = k+1$.

From (3): $\dfrac{BS}{SC} = \dfrac{k+1}{k}$.

From (1): $\dfrac{C'P}{PC} = \dfrac{BS}{SC}$, meaning that $k+1 = \dfrac{k+1}{k}$.

Therefore, $k = 1$, making M and N the midpoints of BC and DD', respectively.

Problem 57A. Proposed by Eliza Andreea Radu

Consider the tetrahedron $VABC$ with a volume equal to 4 such that $m(\measuredangle ACB) = 45^\circ$ and $\frac{AC+3\sqrt{2}(BC+VB)}{\sqrt{18}} = 6$. Find the distance from B to the plane (VAC).

Problem 58A. Proposed by Aida Dragomirescu

Find $a, b, c \in \mathbb{R}$ with the property that $(3a-10)^2 + (3b-10)^2 + (3c-10)^2 + 150 \leq 3\,(ab + ac + bc)$.

Solution Problem 58A

Expanding the left side of the inequality, we get

$$9\left(a^2+b^2+c^2\right)-60\,(a+b+c)+450 \leq 3\,(ab+ac+bc)\,.$$

Dividing both sides by 3, we get

$$3\left(a^2+b^2+c^2\right)-20\,(a+b+c)+150 \leq ab+ac+bc.$$

Multiplying by 2 and then rearranging, we have

$$6\left(a^2+b^2+c^2\right)-40\,(a+b+c)+300-2\,(ab+ac+bc) \leq 0.$$

We now work to form squares:

$$4\left(a^2+b^2+c^2\right)-40\,(a+b+c)+300+2\left(a^2+b^2+c^2\right)-2\,(ab+ac+bc) \leq 0$$

$$\Updownarrow$$

$$\left(4a^2-40a+100\right)+\left(4b^2-40b+100\right)+\left(4c^2-40c+100\right)+(a-b)^2+(b-c)^2+(c-a)^2 \leq 0$$

$$\Updownarrow$$

$$(2a-10)^2+(2b-10)^2+(2c-10)^2+(a-b)^2+(b-c)^2+(c-a)^2 \leq 0.$$

By Trivial Inequality, we know that

$$(2a-10)^2+(2b-10)^2+(2c-10)^2+(a-b)^2+(b-c)^2+(c-a)^2 = 0,$$

which gives us $a=b=c=\frac{10}{2}=5$.

Problem 59A. Proposed by Irina Daria Avram Popa

Find the area of a triangle with sides a, b, c knowing that

$$(3a-b+c)^2+(3b-c+a)^2+(3c-a+b)^2+\frac{1}{3} \leq 2\,(a+b+c)\,.$$

Solution Problem 59A

Let:

$$\begin{aligned} x &= 3a-b+c, \\ y &= 3b-c+a, \\ z &= 3c-a+b. \end{aligned}$$

Adding up the three equations, we get that

$$x+y+z=(3a-b+c)+(3b-c+a)+(3c-a+b)=3\,(a+b+c)\,.$$

We are given that

$$x^2 + y^2 + z^2 + \frac{1}{3} \leq \frac{2}{3}\left(x + y + z\right).$$

Completing the square, our inequality becomes

$$\left(x^2 - \frac{2x}{3} + \frac{1}{9}\right) + \left(y^2 - \frac{2y}{3} + \frac{1}{9}\right) + \left(z^2 - \frac{2z}{3} + \frac{1}{9}\right) \leq 0,$$

$$\Downarrow$$

$$\left(x - \frac{1}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 \leq 0.$$

By Trivial Inequality, we know that $\left(x - \frac{1}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = 0.$
Thus, $x = y = z = \frac{1}{3}$.
We now have that:

$$\begin{cases} 3a - b + c = \dfrac{1}{3}, & (1) \\ 3b - c + a = \dfrac{1}{3}, & (2) \\ 3c - a + b = \dfrac{1}{3}. & (3) \end{cases}$$

$$\text{From } (1) + (2) : 4a + 2b = \frac{2}{3} \Rightarrow b = \frac{1}{3} - 2a. \quad (4)$$

$$\text{From } (1) + (3) : 2a + 4c = \frac{2}{3} \Rightarrow c = \frac{1}{6} - \frac{a}{2}. \quad (5)$$

Substituting (4) and (5) into (3), we get

$$3\left(\frac{1}{6} - \frac{a}{2}\right) - a + \left(\frac{1}{3} - 2a\right) = \frac{1}{3}$$

$$\Downarrow$$

$$\frac{1}{2} - \frac{3a}{2} + \frac{1}{3} - 3a = \frac{1}{3}$$

$$\Downarrow$$

$$\frac{1}{2} = 3a + \frac{3a}{2}$$

$$\Downarrow$$

$$1 = 9a.$$

Thus, $a = \frac{1}{9}$.

We therefore have that

$$b = \frac{1}{3} - 2a = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}.$$

In addition, we get that

$$c = \frac{1}{6} - \frac{a}{2} = \frac{1}{6} - \frac{1}{18} = \frac{2}{18} = \frac{1}{9}.$$

Since $a = b = c = \dfrac{1}{9}$, we know that $\triangle ABC$ is equilateral.

We can now calculate the area of $\triangle ABC$:

$$A = \frac{a^2\sqrt{3}}{4} = \frac{\left(\frac{1}{9}\right)^2 \cdot \sqrt{3}}{4} = \frac{\sqrt{3}}{81} \cdot \frac{1}{4} = \frac{\sqrt{3}}{324}.$$

Problem 60A. Proposed by Daniel Alexandru Guba

If $a, b, c \in [5, \infty)$, prove that $\dfrac{a+b}{ab+4c+15} + \dfrac{a+c}{ac+4b+15} + \dfrac{b+c}{bc+4a+15} \le \dfrac{1}{2}$.

Solution Problem 60A

Since $a \ge 5$ and $b \ge 5$, we have that $(a-4)(b-4) \ge 1 \Rightarrow ab - 4a - 4b + 16 \ge 1$. Rearranging and adding $4a + 4b + 4c - 1$ both both sides of the inequality, we get

$$ab + 4c + 15 \ge 4a + 4b + 4c.$$

Taking their reciprocals and multiplying both sides of the inequality by $a + b$ gives us

$$\frac{a+b}{ab+4c+15} \le \frac{a+b}{4(a+b+c)}. \tag{1}$$

Similarly,

$$\frac{a+c}{ac+4b+15} \le \frac{a+c}{4(a+b+c)}, \tag{2}$$

$$\frac{b+c}{bc+4a+15} \le \frac{b+c}{4(a+b+c)}. \tag{3}$$

Calculating (1)+(2)+(3), we get

$$\frac{a+b}{ab+4c+15} + \frac{a+c}{ac+4b+15} + \frac{b+c}{bc+4a+15} \le \frac{2(a+b+c)}{4(a+b+c)} = \frac{1}{2}.$$

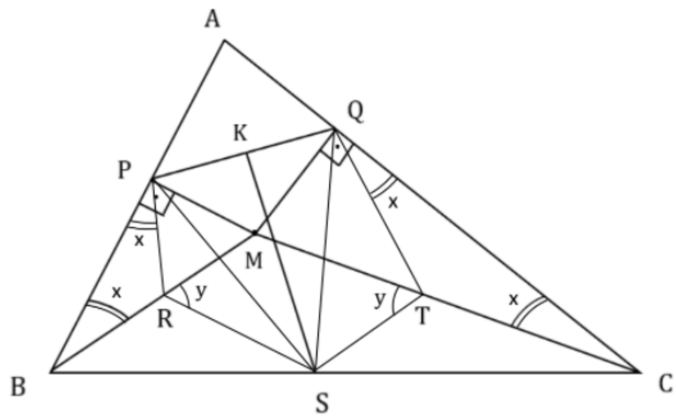
The equality case occurs when $a = b = c = 5$, which gives us

$$\frac{a+b}{ab+4c+15} + \frac{a+c}{ac+4b+15} + \frac{b+c}{bc+4a+15} = \frac{10}{25+20+15} \cdot 3 = \frac{30}{60} = \frac{1}{2}.$$

Problem 61A. Proposed by Gabriel Crisan

Let M be a point in the interior of triangle ABC such that angles ABM and ACM are congruent. From M, build MP perpendicular to side AB, where $P \in AB$, and build MQ perpendicular to side AC, where $Q \in AC$. If S and K are the midpoints of BC and PQ, respectively, prove that SK is perpendicular to PQ.

Solution Problem 61A



Let R be the midpoint of BM and T be the midpoint of CM. Therefore, RS and TS are midlines in $\triangle MBC$. Thus, $RS \parallel MT$ and $TS \parallel MR$. Consequently, $MRST$ is a parallelogram and $m(\measuredangle MRS) = m(\measuredangle MTS) = y$ (see figure). In addition, $RS = \frac{MC}{2}$ and $ST = \frac{MB}{2}$.

In right triangle MPB, PR is the median and $BR = RM = PR$. Thus, $PR = \frac{MB}{2}$. Similarly, QT is a median in right triangle MQC, which gives us that $QT = \frac{MC}{2}$. From the above relationships we can conclude that $RS = MT = QT$ and $PR = RM = ST$. (1)

Let $m(\measuredangle ABM) = m(\measuredangle ACM) = x$. From $PR = BR = \frac{BM}{2}$, $\triangle RPB$ is isosceles and $m(\measuredangle BPR) = m(\measuredangle RBP) = x$. By Exterior Angle Theorem, we have that $m(\measuredangle MRP) = 2x$. Similarly, $m(\measuredangle MTQ) = 2x$ via Exterior Angle Theorem for triangle QTC.

We thus have that

$$m(\measuredangle PRS) = m(\measuredangle MRP) + m(\measuredangle MRS) = 2x + y,$$

$$m(\measuredangle QTS) = m(\measuredangle MTQ) + m(\measuredangle MTS) = 2x + y.$$

From the above relationships, $m(\measuredangle PRS) = m(\measuredangle QTS)$. (2)

From (1) and (2), we can conclude that $\triangle RPS \equiv \triangle TSQ$ by SAS congruence. Therefore, $PS = SQ$, meaning that $\triangle SPQ$ is isosceles. Since SK is the median of isosceles triangle SPQ, it must also be the altitude. Hence, $SK \perp PQ$.

Problem 62A. Proposed by Andrei Croitoru

Let a_1, a_2 . . .a_n be a sequence and $n \in \mathbb{N}$. Knowing that $a_1 = 14$ and

$$a_{n+1} = \frac{(n+3) \cdot a_n + 26}{n+1} \qquad \forall n \geq 1,$$

find all numbers n such that $a_n \in \mathbb{N}$.

Solution Problem 62A

We define a new sequence where $b_n = a_n + 13$. This means that $a_n = b_n - 13$. Thus,

$$b_{n+1} = a_{n+1} + 13.$$

Substituting the recurrence for a_{n+1}, we get

$$b_{n+1} = \frac{(n+3) \cdot a_n + 26}{n+1} + 13.$$

Substituting $a_n = b_n - 13$ into the equation, we have

$$b_{n+1} = \frac{(n+3) \cdot (b_n - 13) + 26 + 13 \cdot (n+1)}{n+1}.$$

Multiplying both sides by $n + 1$ and distributing gives us

$$(n+1) \cdot b_{n+1} = (n+3) \cdot b_n - (n+3) \cdot 13 + 26 + 13 \cdot (n+1).$$

This becomes $(n + 1) \cdot b_{n+1} = (n + 3) \cdot b_n$. Rearranging the equation gives us

$$b_{n+1} = \frac{n+3}{n+1} \cdot b_n.$$

Writing out the first few terms, we get

$$b_1 = a_n + 13 = 27,$$

$$b_2 = \frac{4}{2} \cdot b_1,$$

$$b_3 = \frac{5}{3} \cdot b_2,$$

$$b_4 = \frac{6}{4} \cdot b_3.$$

Multiplying the terms in the sequence, we have

$$b_{n+1} = \frac{4}{2} \cdot \frac{5}{3} \cdot \frac{6}{4} \cdot ... \cdot \frac{n+3}{n+1} \cdot b_1.$$

The sequence telescopes and we get

$$b_{n+1} = \frac{(n+2)(n+3)}{2 \cdot 3} \cdot b_1.$$

This means that

$$b_n = \frac{(n+1)(n+2)}{2 \cdot 3} \cdot 27 = \frac{(n+1)(n+2)}{2} \cdot 9.$$

Since $n+1$ and $n+2$ are consecutive integers, we know that their product must be even. Thus, b_n is always an integer. In addition, we notice that b_n increases as n increases.

Because $a_n = b_n - 13$, we see that a_1, a_2 . . .a_n must be an increasing integer sequence. Since $a_1 = 14$, we know that $a_n \in \mathbb{N}$ for all $n \geq 1$.

Problem 63A. Proposed by Eliza Andreea Radu

Prove that $7^{7^{2021}} + 1$ and $7^{7^{2024}} + 50$ are coprime.

Solution Problem 63A

Assume that the given numbers are not coprime.

For simplicity, we let $x = 7^{7^{2021}}$. Our numbers are now $x+1$ and $x^{343} + 50$.

Let $d = \gcd(7^{7^{2021}} + 1, 7^{7^{2024}} + 50) = \gcd(x+1, x^{343} + 50)$. Thus, $d \mid (x^{343} + 50)$ and $d \mid (x+1)$.

Because $x^{343} + 1 = (x+1)(x^{342} - x^{341} + ... - x + 1)$, we have that $d \mid (x^{343} + 1)$. Consequently,

$$d \mid [(x^{343} + 50) - (x^{343} + 1)] \Leftrightarrow d \mid 49.$$

Therefore, $d \in \{1, 7, 49\}$. Since we assumed that $d \neq 1$, we see that it must be a multiple of 7. However, $7^{7^{2021}} + 1$ and $7^{7^{2024}} + 50$ are not divisible by 7, contradiction. Thus, the assumption is false and the numbers are coprime.

Problem 64A. Proposed by Aida Dragomirescu

Solve in $\mathbb{R}$ the following system $\begin{cases} x^4 + 256 = 4y^3 + 64z \\ y^4 + 256 = 4z^3 + 64x \\ z^4 + 256 = 4x^3 + 64y \end{cases}$

Solution Problem 64A

The sum of the three relationships is: $x^4+y^4+z^4+3\cdot 256 = 4(x^3+y^3+z^3)+64(x+y+z)$. Rearranging, we get

$$x^4+y^4+z^4+3\cdot 256-4(x^3+y^3+z^3)-64(x+y+z)=0, \tag{1}$$

which is equivalent to

$$(x^4-4x^3-64x+256)+(y^4-4y^3-64y+256)+(z^4-4z^3-64z+256)=0.$$

Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $f(x)=x^4-4x^3-64x+256$.

Our equation from (1) can be rewritten as

$$f(x)+f(y)+f(z)=0. \tag{2}$$

Factoring $f(x)$, we get

$$\begin{aligned}
f(x) &= x^4-4x^3-64x+256 \\
&= x^3(x-4)-64(x-4) \\
&= (x-4)(x^3-64) \\
&= (x-4)(x-4)(x^2-4x+16) \\
&= (x-4)^2\left[(x^2-4x+4)+12\right] \\
&= (x-4)^2\left[(x+2)^2+12\right].
\end{aligned}$$

Observation (using Trivial Inequality):

$$f(x)=\underbrace{(x-4)^2}_{\geq 0}\underbrace{\left[(x-2)^2+12\right]}_{>0}\Rightarrow f(x)\geq 0.$$

We see that $f(x)=0$ when $x=4$.

Recalling (2), we see that $f(x)+f(y)+f(z)=0$

$$\Updownarrow$$

$$\begin{cases}
(x-4)^2\left[(x-2)^2+12\right]=0\\
(y-4)^2\left[(y-2)^2+12\right]=0\\
(z-4)^2\left[(z-2)^2+12\right]=0
\end{cases}$$

Thus, $x=y=z=4$.

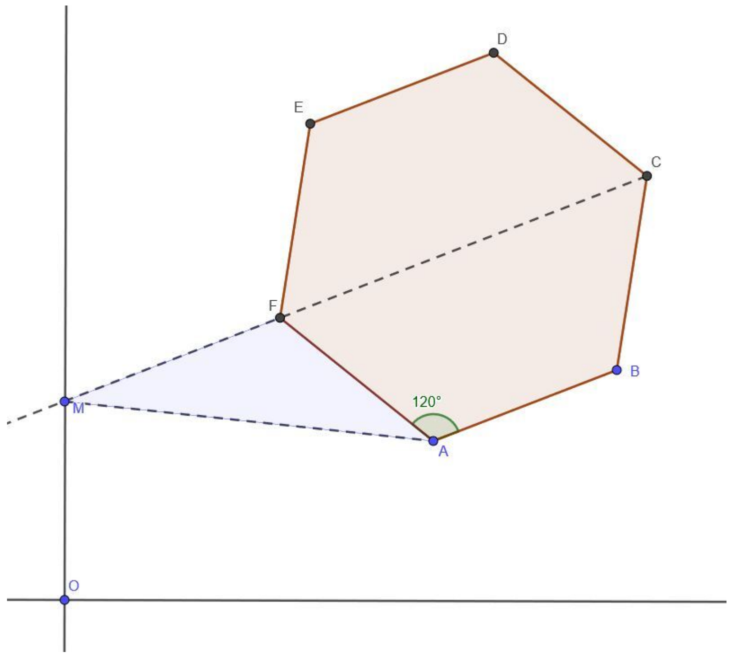
Verification:

$$4^4+256=4\cdot 4^3+64\cdot 4.$$

Problem 65A. Proposed by Daisy Sheng

Regular hexagon $ABCDEF$, with vertices labeled in a counterclockwise order, has coordinates $A(4+\sqrt{3},2)$ and $B(8+\sqrt{3},4)$. Diagonal CF is extended until it intersects the y-axis at M. Find the area of triangle BAM.

Solution Problem 65A



We are given that $ABCDEF$ is a regular hexagon. Thus, each interior angle must be

$$\left(\frac{180\cdot(6-2)}{6}\right)^{\circ}=120^{\circ}.$$

Therefore, $m(\measuredangle BAF)=120^{\circ}$.

We know from the properties of a hexagon that $CF \parallel AB$. Thus, the slope of the line through CF must equal the slope of the line through AB. We see that

$$m_{AB}=\frac{Y_B-Y_A}{X_B-X_A}=\frac{4-2}{(8+\sqrt{3})-(4+\sqrt{3})}=\frac{2}{4}=\frac{1}{2}.$$

In order to get the coordinates of M, we must get the equation of CF. We shall use the complex plane to help us find F. We have that $B=(8+\sqrt{3})+4i$ and $A=(4+\sqrt{3})+2i$. In order to rotate B around A in a 120° angle, we must translate A to the origin and then apply cis.

We let A', B', and F' be the translated versions of A, B and F. We have that $A' = 0$, $B' = (8+\sqrt{3}) - (4+\sqrt{3}) + 4i - 2i = 4 + 2i$, and $F' = (X_F - 4 - \sqrt{3}) + (Y_F - 2)i$. Applying cis, we have that

$$\begin{aligned} F' &= B' cis 120^\circ \\ &= (4+2i)(\cos 120^\circ + i \sin 120^\circ) \\ &= (4+2i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -2 + i \cdot 2\sqrt{3} - i - \sqrt{3} \\ &= (-2-\sqrt{3}) + (2\sqrt{3}-1)i. \end{aligned}$$

We can now find X_F and Y_F, the original coordinates of the point F. We have that

$$X_F - 4 - \sqrt{3} = -2 - \sqrt{3},$$

giving us that $X_F = 2$. We also have that

$$Y_F - 2 = 2\sqrt{3} - 1,$$

which simplifies to $Y_F = 2\sqrt{3} + 1$. Therefore, $F(2, 2\sqrt{3}+1)$.

With the coordinates of F and $m_{CF} = m_{AB} = \frac{1}{2}$, we can now get the equation of CF. The general equation is $y = \frac{1}{2}x + b$. Plugging F in, we get

$$Y_F = \frac{1}{2} X_F + b,$$

which gives us

$$2\sqrt{3} + 1 = \frac{1}{2} \cdot 2 + b.$$

Solving for b quickly yields $b = 2\sqrt{3}$. Because M is the y-intercept of CF, therefore $M(0, 2\sqrt{3})$.

Our coordinates for triangle BAM are thus $A(4+\sqrt{3}, 2), B(8+\sqrt{3}, 4)$, and $M(0, 2\sqrt{3})$.

I will now present two possible methods to finish the problem.

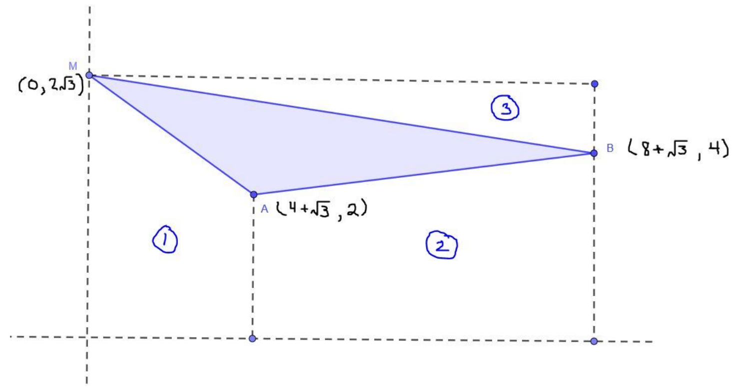
Solution 1 - Shoelace Theorem for Triangle BAM

We have that the area of BAM is

$$\begin{aligned} A &= \frac{1}{2}|(4+\sqrt{3}) \cdot 4 + (8+\sqrt{3}) \cdot 2\sqrt{3} + 0 \cdot 2 - 2 \cdot (8+\sqrt{3}) - 4 \cdot 0 - 2\sqrt{3} \cdot (4+\sqrt{3})| \\ &= \frac{1}{2}|16 + 4\sqrt{3} + 16\sqrt{3} + 6 - 16 - 2\sqrt{3} - 8\sqrt{3} - 6| \\ &= \frac{1}{2}|10\sqrt{3}| \\ &= 5\sqrt{3}. \end{aligned}$$

Therefore, the area of BAM is $5\sqrt{3}$.

Solution 2 - Subtracting Areas from a Rectangle



We see that the area of the large rectangle is $(8+\sqrt{3})\cdot 2\sqrt{3} = 16\sqrt{3}+6$. The area of 1, a trapezoid, is

$$\frac{2+2\sqrt{3}}{2}\cdot(4+\sqrt{3}) = (1+\sqrt{3})(4+\sqrt{3}) = 4+\sqrt{3}+4\sqrt{3}+3 = 7+5\sqrt{3}.$$

The area of 2, also a trapezoid, is

$$\frac{2+4}{2}\cdot(8+\sqrt{3}-4-\sqrt{3}) = 3\cdot 4 = 12.$$

The area of 3, a triangle, is

$$\frac{(2\sqrt{3}-4)(8+\sqrt{3})}{2} = (\sqrt{3}-2)(8+\sqrt{3}) = 8\sqrt{3}+3-16-2\sqrt{3} = 6\sqrt{3}-13.$$

Therefore, the area of BAM is

$$\begin{aligned} A &= (16\sqrt{3}+6)-(7+5\sqrt{3})-12-(6\sqrt{3}-13) \\ &= 16\sqrt{3}-5\sqrt{3}-6\sqrt{3}+6-7-12+13 \\ &= 5\sqrt{3}. \end{aligned}$$

Problem 39B. Proposed by Alexander Monteith-Pistor

For $n \in \mathbb{N}$, let $S(n)$ and $P(n)$ denote the sum and product of the digits of n (respectively). For how many $k \in \mathbb{N}$ do there exist positive integers $n_1, ..., n_k$ satisfying

$$\sum_{i=1}^{k} n_i = 2021$$

$$\sum_{i=1}^{k} S(n_i) = \sum_{i=1}^{k} P(n_i)$$

Problem 40B. Proposed by Vedaant Srivastava

Two identical rows of numbers are written on a chalkboard, each comprised of the natural numbers from 1 to 10! inclusive. Determine the number of ways to pick one number from each row such that the product of the two numbers is divisible by 10!

Problem 42B. Proposed by Andy Kim

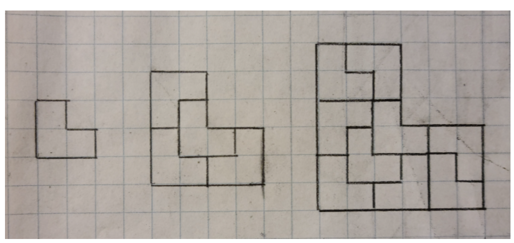
Define an L-region of size n as an L-shaped region with two sides of length $2n$ and four sides of length n, and define an L-tile to be a tile with the same shape as an L-region of size 1 (i.e. a 2×2 square with one 1×1 square missing). Prove that an L-region of size n can be tiled with L-tiles for all positive integers n.

Solution Problem 42B.

We proceed by induction.

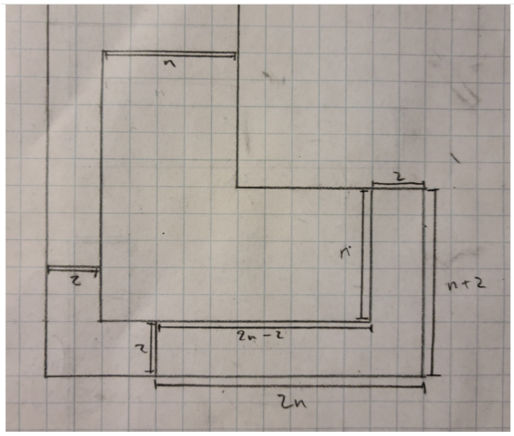
Base case: $n = 1, 2, 3.$

For these cases, the L-regions can be tiled as follows:



Inductive step:

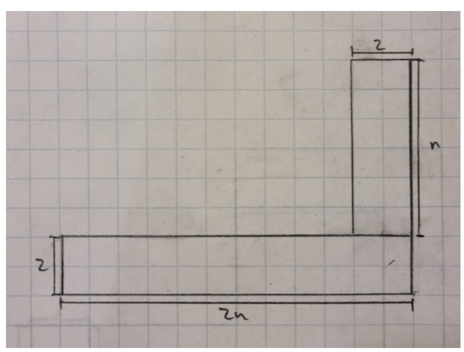
Suppose an L-region of size k is tilable, for some $k \geq 2$. Now, we show an L-region of size $k + 2$ is tilable. We split the L-region of size $k + 2$ as follows:



Since an L-region of size 2 and k are tilable, all that remains to show is that the two irregular L-shaped regions are tilable. We proceed by casework.

Case 1: $n \bmod 3 = 0$

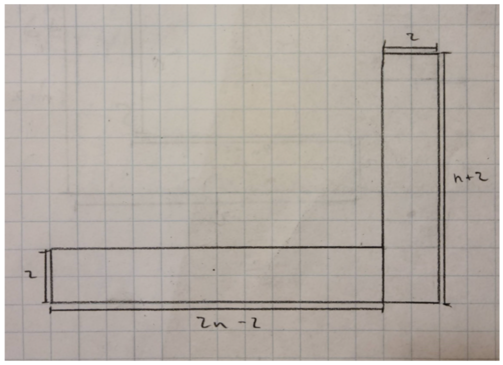
We have $n = 3k$ for some $k \in \mathbb{Z}^+$, noting that $n \geq 2$. Then, the irregular L-shaped region can be split into 2 sections: a rectangle of dimension $2 \times n$, and another of dimension $2 \times 2n$. This split looks like the following:



Substituting $n = 3k$, we have that the two rectangles are of dimension $2 \times 3k$ and $2 \times 6k$. Since a rectangle of dimension 2×3 is clearly tilable with two L-tiles, we have that these two rectangles can also be tiled, by attaching the respective number of 2×3 rectangles side by side. So, the irregular L-shaped region can be tiled, and thus the entire L-region of size $k+2$ can be tiled, for $n \bmod 3 = 0$.

Case 2: $n \bmod 3 = 1$

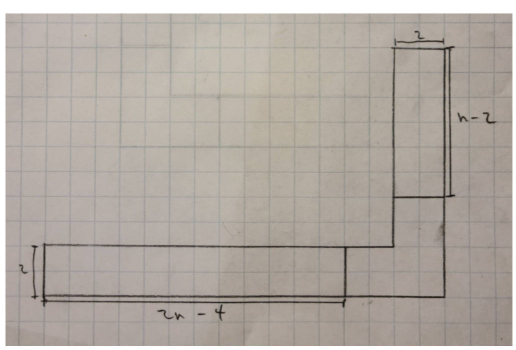
We have $n = 3k+1$ for some $k \in \mathbb{Z}^+$, noting that $n \geq 2$. Then, the irregular L-shaped region can be split into 2 sections: a rectangle of dimension $2 \times (n+2)$, and another of dimension $2 \times (2n-2)$. This split looks like the following:



Substituting $n = 3k + 1$, we have that the two rectangles are of dimension $2 \times 3k + 3$ and $2 \times 6k$. These rectangles are again tilable by attaching the respective number of 2×3 rectangles side by side. So, the irregular L-shaped region can be tiled, and thus the entire L-region of size $k + 2$ can be tiled, for $n \bmod 3 = 1$.

Case 3: $n \bmod 3 = 2$

We have $n = 3k + 2$ for some $k \in \mathbb{Z}^+ \cup \{0\}$, noting that $n \geq 2$. Then, the irregular L-shaped region can be split into 3 sections: an L-region of size 2 in the corner, a rectangle of dimension $2 \times n - 2$, and another of dimension $2 \times 2n - 4$. This split looks like the following:



Substituting $n = 3k + 2$, we have that the two rectangles are of dimension $2 \times 3k$ and $2 \times 6k$. So, they are tilable by attaching the respective number of 2×3 rectangles side by side. Since a L-region of size 2 is tilable, we have that the irregular L-shaped region can be tiled, and thus the entire L-region of size $k + 2$ can be tiled, for $n \bmod 3 = 2$.

From all three cases, we have that an L-region of size $k + 2$ is tilable.

Then, by induction, an L-region of size n is tilable with L-tiles, for all $n \in \mathbb{Z}^+$.

Problem 49B. Proposed by Cosmina Ghitescu

Solve the equation $2021 + 2^x = 7^y 5^z$, where x, y, $z \in \mathbb{N}$.

Solution Problem 49B

We consider cases.

Case 1: $z > 0$
We have that $2021 + 2^x \equiv 7^y 5^z \equiv 0 \pmod 5$. Since $2021 + 2^x$ is odd, the units digit of $2021 + 2^x$ must be 5. Thus, the units digit of 2^x is 4. We know that the units digits of powers of 2 come in repeated cycles (sequence: 2, 4, 8, 6). We discover that 2^x has a units digit of 4 when $x \equiv 2 \pmod 4$. Therefore, x is even.

Taking the equation mod 3, we have that

$$2021 + 2^x \equiv 2 + (-1)^x \equiv 3 \equiv 0 \pmod 3.$$

However, $7^y 5^z$ is never a multiple of 3. Consequently, there are no solutions for this case.

Case 2: $z = 0$
Our equation becomes $2021 + 2^x = 7^y$.

If $y = 0$, we get that $2021 + 2^x = 1$, which has no solutions.

If $y > 0$, we see that $7 \mid (2021 + 2^x)$. (*)

Let us consider cases, where $x = 3k, 3k + 1$ or $3k + 2$ for $k \in \mathbb{N}$.

If $k = 0$, we get that $x = 0$, $x = 1$ or $x = 2$. None of them lead to a valid solution since 2022, 2023, and 2025 are not powers of 7.

We now take a look at the cases where $k \geq 1$.

Case 2.1: $x = 3k$

We have that

$$\begin{aligned}2021 + 2^x &= 2021 + 8^k \\ &= (7 \cdot 288 + 5) + 8^k \\ &\equiv 5 + (1)^k \pmod 7 \\ &\equiv 6 \not\equiv 0. \pmod 7\end{aligned}$$

From (*), this case yields no solutions.

Case 2.2: $x = 3k + 2$

We have that

$$\begin{aligned}2021 + 2^x &= 2021 + 8^k \cdot 4 \\ &= (7 \cdot 288 + 5) + 8^k \cdot 4 \\ &\equiv 5 + (1)^k \cdot 4 \pmod 7 \\ &\equiv 9 \pmod 7 \\ &\equiv 2 \not\equiv 0. \pmod 7\end{aligned}$$

From (*), this case yields no solutions.

Case 2.3: $x = 3k + 1$

We have that

$$\begin{aligned}2021 + 2^x &= 2021 + 8^k \cdot 2 \\ &= (7 \cdot 288 + 5) + 8^k \cdot 2 \\ &\equiv 5 + (1)^k \cdot 2 \pmod 7 \\ &\equiv 7 \pmod 7 \\ &\equiv 0. \pmod 7\end{aligned}$$

We find that (*) is satisfied. Now, we have $2021 + 8^k \cdot 2 = 7^y$.

Taking each side mod 8, we get

$$2021 + 8^k \cdot 2 \equiv 5 \pmod 8,$$

$$7^y \equiv (-1)^y \equiv 1 \text{ or } 7 \pmod 8.$$

Since they are not congruent mod 8, we get no solutions.

To conclude, the equation has no solutions in $\mathbb{N}$.

Problem 52B. Proposed by Daisy Sheng

Find the general form for the integer k such that the expression

$$2^{n+1} + 5^{n+2} \cdot 3^{2n+4} \cdot k + (2k + 1) \cdot (47 \cdot 3)^{n+3}$$

is divisible by 2021 for all positive integers n that are odd multiples of 3. For reference, $2021 = 43 \cdot 47$.

Solution Problem 52B

Simplifying our expression, we get

$$\begin{aligned}&2^{n+1}+(5\cdot 3^2)^{n+2}\cdot k+(2k+1)\cdot(47\cdot 3)^{n+3}\\ =\ &2^{n+1}+(45)^{n+2}\cdot k+(2k+1)\cdot(47\cdot 3)^{n+3}.\end{aligned}$$

We know that $2021 = 43\cdot 47$. Because 43 and 47 are relatively prime, we consider their mods separately.

We see that $45 \equiv 2 \pmod{43}$ and $47\cdot 3 \equiv 4\cdot 3 \equiv 12 \pmod{43}$. We would like to make the residue of 12 smaller, so we try squaring. We get that $12^2 \equiv 144 \equiv 43\cdot 3+15 \equiv 15 \pmod{43}$. This is still large, so we try cubing to get $12^3 \equiv 12^2\cdot 12 \equiv 15\cdot 12 \equiv 180 \equiv 43\cdot 4+8 \equiv 8 \pmod{43}$. We also note that $12^y \equiv 2^y \pmod{43}$ when y is a positive integer multiple of 3 because we can divide the 12^3 into whole number groups to get $8 \pmod{43}$. Thus, we can properly apply the cubing as $n+3$ is a multiple of 3.

In mod 43, our expression thus becomes

$$\begin{aligned}2^{n+1}+45^{n+2}\cdot k+(2k+1)\cdot(47\cdot 3)^{n+3} &\equiv 2^{n+1}+2^{n+2}\cdot k+(2k+1)\cdot 12^{n+3} \pmod{43}\\ &\equiv 2^{n+1}(1+2k)+(2k+1)\cdot(12^3)^{\frac{n+3}{3}}\\ &\equiv 2^{n+1}(2k+1)+(2k+1)\cdot(8)^{\frac{n+3}{3}}\\ &\equiv 2^{n+1}(2k+1)+(2k+1)\cdot(2^3)^{\frac{n+3}{3}}\\ &\equiv 2^{n+1}(2k+1)+(2k+1)\cdot(2)^{n+3}\\ &\equiv (2k+1)(2^{n+1}+2^{n+3})\\ &\equiv 5\cdot(2k+1)\cdot 2^{n+1}.\end{aligned}$$

Because $2, 5 \not\equiv 0 \pmod{43}$, therefore $2k+1 \equiv 0 \pmod{43}$ if our expression is to be divisible by 43. Rearranging, we get that $2k \equiv -1 \pmod{43}$. We easily identify 22 to be the multiplicative inverse since $2\cdot 22 = 44$. We thus get that $k \equiv -22 \equiv 21 \pmod{43}$.

We now consider the expression mod 47. We know from the factorization given in the problem that $141 \equiv 0 \pmod{47}$. In addition, $n+2$ is odd because n is odd. We thus get that

$$\begin{aligned}2^{n+1}+45^{n+2}\cdot k+(2k+1)\cdot(47\cdot 3)^{n+3} &\equiv 2^{n+1}+(-2)^{n+2}\cdot k+0 \pmod{47}\\ &\equiv 2^{n+1}-2^{n+2}\cdot k\\ &\equiv 2^{n+1}(1-2k).\end{aligned}$$

Because $2 \not\equiv 0 \pmod{47}$, therefore $1-2k \equiv 0 \pmod{47}$. Solving, we get $2k \equiv 1 \pmod{47}$. We see that $\frac{48}{2} = 24$ is the multiplicative inverse. We therefore get that $k \equiv 24 \pmod{47}$.

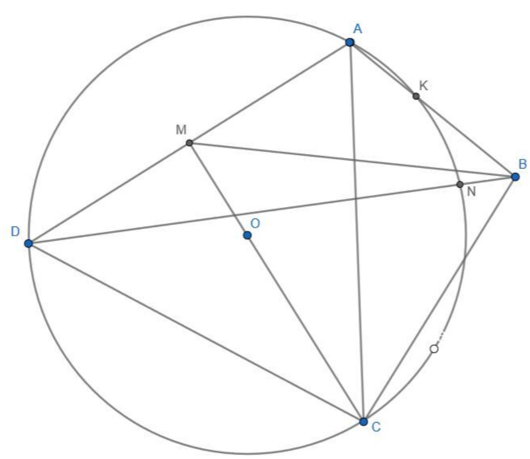
We now solve the system of congruence, $k \equiv 21 \pmod{43}$ and $k \equiv 24 \pmod{47}$. We let $k = 47b + 24 \quad (*)$, for some integer b. Substituting this into the second congruence, we get

$$47b + 24 \equiv 21 \pmod{43}.$$

Simplifying gives us $4b \equiv -3 \pmod{43}$. The multiplicative inverse of 4 is $\frac{44}{4} = 11$. Thus, we get $b \equiv -33 \equiv 10 \pmod{43}$. We let $b = 43a + 10$, for some integer a. Substituting this back into (*), we get $k = 47(43a + 10) + 24 = 2021a + 470 + 24 = 2021a + 494$, for some integer a.

Problem 53B. Proposed by Daisy Sheng

Quadrilateral $ABCD$ is constructed with M as the midpoint of AD and $2AB > AD$. Let the circumcircle of triangle ACD intersect AB at K and BD at N, where arc $KN = 30°$ and $\measuredangle ABD = 45°$ (see figure below). If $CD^2 + 2AC \cdot AB = 4AB^2 + AC \cdot CD$ and $\cos(m(\measuredangle BAD)) = \frac{AB - AC}{AD}$, prove that $BM = CM$.
**Inspired by a TST Problem for Girls' Math Team Canada.*



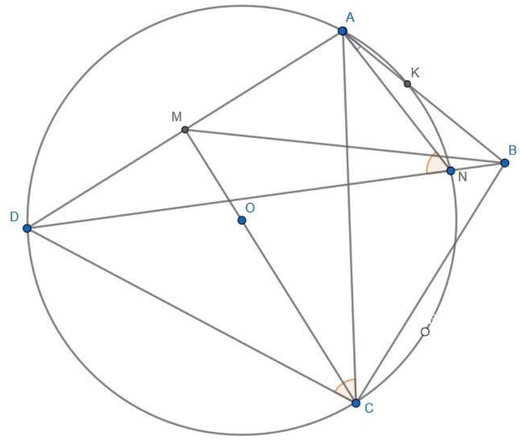
Solution Problem 53B

Connect A and N. Because the length of arc KN is $30°$, we see that $m(\measuredangle KAN) = 15°$ by the Inscribed Angle Theorem. Thus by Exterior Angle Theorem,

$$m(\measuredangle AND) = m(\measuredangle KAN) + m(\measuredangle ABD) = 15° + 45° = 60°.$$

Because $\measuredangle AND$ and $\measuredangle ACD$ both subtend the same arc, therefore

$$m(\measuredangle ACD) = m(\measuredangle AND) = 60°.$$



Because M is the midpoint of AD, therefore CM is a median in triangle ACD. By Median Theorem, a special case of Stewart's Theorem, we get that

$$CM^2 = \frac{2AC^2 + 2CD^2 - AD^2}{4}.$$

We are given that

$$CD^2 + 2AC \cdot AB = 4AB^2 + AC \cdot CD.$$

Applying difference of squares, we get

$$(CD - 2AB)(CD + 2AB) = AC(CD - 2AB).$$

We take out a common factor of $CD - 2AB$ to get

$$(CD - 2AB)(CD + 2AB - AC) = 0.$$

By Triangle Inequality in triangle ADC, we have that $AD + CD > AC$. Combining this with $2AB > AD$, which is given, we have that

$$2AB + CD > AD + CD > AC.$$

Thus, $2AB + CD - AC > 0$. Because $(CD - 2AB)(CD + 2AB - AC) = 0$ and $CD + 2AB - AC > 0$, we see that $CD - 2AB = 0$. Therefore, $CD = 2AB$.

I will now present two possible methods to finish the problem.

Solution 1 - Cosine Law in Triangle ABM

Because M is the midpoint, we see that $AD = 2AM$. By Cosine Law in Triangle ABM, we get that

$$\begin{aligned}
BM^2 &= AM^2 + AB^2 - 2AM \cdot AB \cdot \cos(m(\measuredangle MAB)) \\
&= AM^2 + AB^2 - 2AM \cdot AB \cdot \cos(m(\measuredangle DAB)) \\
&= AM^2 + AB^2 - 2AM \cdot AB \cdot \frac{AB - AC}{AD} \\
&= AM^2 + AB^2 - AB^2 + AB \cdot AC \\
&= \frac{AD^2}{4} + AB \cdot AC \\
&= \frac{AD^2 + 4AB \cdot AC}{4} \\
&= \frac{AD^2 + 2CD \cdot AC}{4}
\end{aligned}$$

By Cosine Law in triangle ADC, we get that

$$\begin{aligned}
AD^2 &= AC^2 + CD^2 - 2AC \cdot CD \cdot \cos(m(\measuredangle ACD)) \\
&= AC^2 + CD^2 - 2AC \cdot CD \cdot \cos 60^\circ \\
&= AC^2 + CD^2 - AC \cdot CD
\end{aligned}$$

This is equivalent to $2AC^2 + 2CD^2 = 2AD^2 + 2AC \cdot CD$. We see that

$$CM^2 = \frac{2AC^2 + 2CD^2 - AD^2}{4} = \frac{AD^2 + 2AC \cdot CD}{4} = BM^2.$$

Because $CM, BM > 0$, we have thus proved that $CM = BM$.

Solution 2 - Median Theorem in Triangle ABD

Because M is the midpoint, BM is thus the median of triangle ABD. By Median Theorem in Triangle ABD, we get that

$$BM^2 = \frac{2AB^2 + 2BD^2 - AD^2}{4}.$$

If BM and CM are equal, we must show that $2AB^2 + 2BD^2 = 2AC^2 + 2CD^2$. From Cosine Law in Triangle ADC (see Solution 1), we have

$$AD^2 = AC^2 + CD^2 - AC \cdot DC.$$

This is equivalent to $2AD^2 + 2AC \cdot DC = 2AC^2 + 2CD^2 \quad (*)$. From Cosine Law in Triangle ABD, we get

$$BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cdot \cos(m(\measuredangle BAD)).$$

Substituting $\cos(m(\measuredangle BAD)) = \frac{AB-AC}{AD}$ into the above equation, we have that

$$BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cdot \frac{AB - AC}{AD},$$

which becomes

$$BD^2 = AB^2 + AD^2 - 2AB^2 + 2AB \cdot AC.$$

We see that

$$\begin{aligned} 2AB^2 + 2BD^2 &= 2AB^2 + 2 \cdot (AB^2 + AD^2 - 2AB^2 + 2AB \cdot AC) \\ &= 2AB^2 + 2 \cdot (AD^2 - AB^2 + 2AB \cdot AC) \\ &= 2AD^2 + 4AB \cdot AC \\ &= 2AD^2 + 2CD \cdot AC \\ &= 2AC^2 + 2CD^2, \end{aligned}$$

with the last equality coming from (*). Thus, we have that $BM^2 = CM^2$, proving that $BM = CM$.

Problem 54B. Proposed by Max Jiang

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following property:

Given an ordered pair $(x, y) \in \mathbb{R}^2$, we have either

$$\begin{gathered} f(x) - f(y) = f(x^2 - y^2) \\ f(x)f(y) = f(x^2y^2) \end{gathered}$$

or

$$\begin{gathered} f(y) \neq 0 \\ f(x) + f(y) = f(x^2 - y^2) \\ f(x)/f(y) = f(x^2y^2). \end{gathered}$$

Note: it is possible that an ordered pair $(x, y), x \neq y$ satisfies the first set of conditions while the ordered pair (y, x) satisfies the other set.

Solution Problem 54B

Call the conditions

$$\begin{gathered} f(x) - f(y) = f(x^2 - y^2) \\ f(x)f(y) = f(x^2y^2) \end{gathered}$$

(1) and the conditions

$$\begin{gathered} f(y) \neq 0 \\ f(x) + f(y) = f(x^2 - y^2) \\ f(x)/f(y) = f(x^2y^2). \end{gathered}$$

(2).

Consider the ordered pair $(0, 0)$. If we have (2), then

$$\begin{aligned} f(0) + f(0) &= f(0^2 - 0^2) \\ \implies f(0) &= 0, \end{aligned}$$

which contradicts that $f(y) \neq 0$. Thus, we must have (1), so

$$\begin{aligned} f(0) - f(0) &= f(0^2 - 0^2) \\ \implies f(0) &= 0. \end{aligned}$$

Now consider (r, r) for some $r \in \mathbb{R}$. If we have (2), then

$$\begin{aligned} f(r) + f(r) &= f(r^2 - r^2) \\ \implies f(r) &= 0, \end{aligned}$$

which contradicts that $f(y) \neq 0$. Thus, we must have (1), so

$$\begin{aligned} f(r)f(r) &= f(r^2 r^2) \\ \implies f(r^4) &= f(r)^2 && \geq 0. \end{aligned}$$

Since $r \in \mathbb{R}$ was arbitrary and $r^4 \geq 0, \forall r \in \mathbb{R}$, we have $f(x) \geq 0, \forall x \geq 0$.

Consider the pair $(r, -r)$ for some $r \in \mathbb{R}$. If we have (2), then

$$\begin{aligned} f(r) + f(-r) &= f(r^2 - (-r)^2) \\ \implies f(r) &= -f(-r) \\ f(r)/f(-r) &= f(r^2(-r)^2) \\ \implies f(r^4) &= -1. \end{aligned}$$

However, since $r^4 \geq 0$ we must have $f(r^4) \geq 0$. This is a contradiction so we must have (1). This gives

$$\begin{aligned} f(r) - f(-r) &= f(r^2 - (-r)^2) \\ \implies f(r) &= f(-r). \end{aligned}$$

In particular, we have $f(-r) = f(r) \geq 0, \forall r \geq 0$ so $f(x) \geq 0, \forall x \in \mathbb{R}$.

We now prove that if $f(b) = 0$ for some $b \neq 0$, we must have $f(x) = 0, \forall x \in \mathbb{R}$.

For such a b, the ordered pair (a, b) must satisfy (1) for any $a \in \mathbb{R}$ since $f(b) = 0$. Then, we have

$$\begin{aligned} f(a^2 b^2) &= f(a)f(b) \\ &= 0 \end{aligned}$$

Since $b \neq 0$, as a ranges over the reals, a^2b^2 can take on any nonnegative value. Thus, we have $f(x) = 0, \forall x \geq 0$. Then, $f(x) = f(-x), \forall x \geq 0$ so $f(-x) = 0, \forall x \geq 0$ as well. This completes the proof.

Consider an ordered pair $(a, b), a \geq b$ with $a, b \neq 0$ that satisfies (1). We have

$$\begin{aligned} f(a) - f(b) &= f(a^2 - b^2) \\ &\geq 0 \end{aligned}$$

If $f(a^2 - b^2) = 0$ we immediately have $f(x), \forall x \in \mathbb{R}$ from our previous result. Otherwise, $f(a) - f(b) > 0 \implies f(b) - f(a) < 0$.

Then, for the ordered pair (b, a), we see that it cannot satisfy (1) since this would give $f(b^2 - a^2) = f(b) - f(a) < 0$. Thus, it must satisfy (2), which gives

$$\begin{aligned} f(b) + f(a) &= f(b^2 - a^2) \\ &= f(a^2 - b^2) && (f(x) = f(-x)) \\ &= f(a) - f(b) \\ \implies f(b) &= 0 \\ \implies f(x) &= 0, \forall x \in \mathbb{R}. \end{aligned}$$

Otherwise, we have that (a, b) satisfies (2) for any $a, b \in \mathbb{R}, a, b \neq 0$. We have

$$f(a)/f(b) = f(a^2b^2).$$

Then, the pair (b, a) satisfies (2), so we have

$$\begin{aligned} f(b)/f(a) &= f(b^2a^2) \\ &= f(a^2b^2) \\ &= f(a)/f(b) \\ \implies f(a)^2 &= f(b)^2 \\ \implies f(a) &= f(b) && (f(x) \geq 0, \forall x \in \mathbb{R}) \\ \implies f(a^2b^2) &= 1, \forall a, b \in \mathbb{R}, a, b \neq 0. \end{aligned}$$

As a and b range over the positive reals, a^2b^2 can take on any positive value, so we have $f(x) = 1, \forall x > 0$. Then $f(x) = f(-x), \forall x \in \mathbb{R}$ so we have $f(x) = 1, \forall x \in \mathbb{R}$.

However, if this holds, for any $a, b \in \mathbb{R}, a, b \neq 0, a \neq b$ we have

$$\begin{aligned} f(a) + f(b) &= f(a^2 - b^2) \\ 1 + 1 &= 1 \\ \implies 2 &= 1, \end{aligned}$$

which is a contradiction. Thus, this case cannot happen.

Finally, this means that the only function that satisfies the condition is

$$f(x) = 0, \forall x \in \mathbb{R}.$$

Problem 56B. Proposed by Alexander Monteith-Pistor

A game is played with white and black pieces and a chessboard (8 by 8). There is an unlimited number of identical black pieces and identical white pieces. To obtain a starting position, any number of black pieces are placed on one half of the board and any number of white pieces are placed on the other half (at most one piece per square). A piece is called matched if its color is the same of the square it is on. If a piece is not matched then it is mismatched. How many starting positions satisfy the following condition

$$\# \text{ of matched pieces} - \# \text{ of mismatched pieces} = 16$$

(your answer should be a binomial coefficient)

Problem 58B. Proposed by Aida Dragomirescu

Solve in $\mathbb{R}$ the system consisting of the following equations

$$\sum_{i=1}^{4} x_i^4 = 4\left(\sum_{i=1}^{4} x_i^3\right) - 108 \tag{1}$$

$$\sum_{i=1}^{4} x_i^2 = 6\left(\sum_{i=1}^{4} x_i\right) - 36 \tag{2}$$

Solution Problem 58B

Multiplying the second equation by 2, we get

$$2\sum_{i=1}^{4} x_i^2 = 12\left(\sum_{i=1}^{4} x_i\right) - 72. \tag{3}$$

Adding equations (1) and (3) results in:

$$\sum_{i=1}^{4} x_i^4 + 2\sum_{i=1}^{4} x_i^2 = 4\left(\sum_{i=1}^{4} x_i^3\right) - 108 + 12\left(\sum_{i=1}^{4} x_i\right) - 72.$$

Rearranging the above equation, we have

$$\sum_{i=1}^{4} x_i^4 - 4\left(\sum_{i=1}^{4} x_i^3\right) + 2\sum_{i=1}^{4} x_i^2 - 12\left(\sum_{i=1}^{4} x_i\right) + 180 = 0. \tag{4}$$

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^4 - 4x^3 + 2x^2 - 12x + 45$.

Equation (4) can be rewritten as:

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) = 0.$$

Factoring $f(x)$ gives us $f(x) = (x-3)^2\left(x^2+2x+5\right)$. This can be found through Rational Root Theorem and polynomial division.

Observation: $f(x) \geq 0,\ \forall x \in \mathbb{R}$.
We know that the observation is true since $(x-3)^2 \geq 0$ via Trivial Inequality and $x^2 + 2x + 5 > 0$ since it has a negative discriminant and a positive leading coefficient.

Equality is reached when $x = 3$.

So, $\sum_{i=1}^{4} f(x_i) = 0$ when $x_1 = x_2 = x_3 = x_4 = 3$ (unique solution).

Problem 59B. Proposed by Alexandru Benescu

Prove that

$$\text{i) } \frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq ab + bc + ca$$

$$\text{ii) } \frac{a^{x+y+1}}{b^x} + \frac{b^{x+y+1}}{c^x} + \frac{c^{x+y+1}}{a^x} \geq a^y b + b^y c + c^y a$$

where $a, b, c \in \mathbb{R}_+$ and $x, y \in \mathbb{N}$.

Solution Problem 59B

Part i):

By AM-GM, we know that $a^4 + b^4 + b^4 + b^4 \geq 4ab^3$. Rearranging, we get that $a^4 \geq 4ab^3 - 3b^4$. Dividing both sides by b^3 gives us $\frac{a^4}{b^3} \geq 4a - 3b$. Multiplying both sides by a results in

$$\frac{a^5}{b^3} \geq 4a^2 - 3ab.$$

Thus,

$$\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq 4(a^2 + b^2 + c^2) - 3(ab + bc + ca).$$

We know that $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$, which gives us that $a^2 + b^2 + c^2 \geq ab + bc + ac$. So,

$$4(a^2 + b^2 + c^2) - 3(ab + bc + ca) \geq ab + bc + ca.$$

Therefore, $\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq ab + bc + ca$.

Part ii):

We use an approach similar to that in Part i).

By AM-GM, we know that

$$a^{x+1} + \underbrace{b^{x+1} + \cdots + b^{x+1}}_{x \text{ terms}} \geq (x+1)ab^x.$$

Rearranging the inequality, we get

$$a^{x+1} \geq (x+1)ab^x - xb^{x+1}.$$

Dividing both sides by b^x gives us

$$\frac{a^{x+1}}{b^x} \geq (x+1)a - xb.$$

Multiplying both sides of the inequality by a^y yields

$$\frac{a^{x+y+1}}{b^x} \geq (x+1)a^{y+1} - xa^y b.$$

Thus,

$$\frac{a^{x+y+1}}{b^x} + \frac{b^{x+y+1}}{c^x} + \frac{c^{x+y+1}}{a^x} \geq (x+1)(a^{y+1} + b^{y+1} + c^{y+1}) - x(a^y b + b^y c + c^y a).$$

We can assume that $a \geq b \geq c$. This also means that $a^y \geq b^y \geq c^y$. By Rearrangement Inequality, we get that

$$a^{y+1} + b^{y+1} + c^{y+1} \geq a^y b + b^y c + c^y a.$$

Therefore,

$$(x+1)(a^{y+1} + b^{y+1} + c^{y+1}) - x(a^y b + b^y c + c^y a) \geq a^y b + b^y c + c^y a.$$

Hence, we can conclude that

$$\frac{a^{x+y+1}}{b^x} + \frac{b^{x+y+1}}{c^x} + \frac{c^{x+y+1}}{a^x} \geq a^y b + b^y c + c^y a.$$

Problem 60B. Proposed by Cosmina Ghitescu

Consider the triangle ABC with sides $a > b > c$ satisfying the condition

$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

Prove that

$$\frac{3\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < \frac{8R^2}{S}.$$

Solution Problem 60B

We know from the Extended Law of Sines that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Our equality becomes:

$$\frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2} = 2,$$

which rearranges as

$$a^2 + b^2 + c^2 = 8R^2.$$

Letting S be the area of the triangle, we have from above that

$$\frac{8R^2}{S} = \frac{a^2 + b^2 + c^2}{S}. \tag{1}$$

We will now prove that our triangle is right-angled.

Method 1: Original

We know that $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, where O is the circumcenter and H is the orthocenter. For our triangle ABC, we have that

$$OH^2 = 9R^2 - 8R^2 = R^2 \Rightarrow OH = R.$$

The orthocenter only lies on the circumcircle when triangle ABC is a right-angled triangle. Since we are given that $a > b > c$, we conclude that A is the 90 degree angle.

Method 2: Presented by AEs

We can also use trigonometry. We have that

$$\begin{aligned} 2 &= \sin^2 A + \sin^2 B + \sin^2 C \\ &= \sin^2 A + \sin^2 B + \sin^2(180 - (A + B)) \\ &= \sin^2 A + \sin^2 B + \sin^2(A + B). \end{aligned}$$

Through the Pythagorean Identity and rearranging, we get

$$\sin^2(A + B) = \cos^2 A + \cos^2 B,$$

which expands to give us

$$\sin^2 A \cos^2 B + \cos^2 A \sin^2 B + 2 \sin A \cos B \cos A \sin B = \cos^2 A + \cos^2 B.$$

Factoring and applying Pythagorean Identity once again, we get

$$2 \sin A \cos B \cos A \sin B = \cos^2 A \cos^2 B + \cos^2 B \cos^2 A,$$

which simplifies to

$$\sin A \cos B \cos A \sin B = \cos^2 A \cos^2 B.$$

Rearranging and factoring gives us

$$\begin{aligned}
0 &= \sin A \cos B \cos A \sin B - \cos^2 A \cos^2 B \\
&= \cos B \cos A(\sin A \sin B - \cos A \cos B) \\
&= \cos B \cos A \cdot [-\cos(A+B)] \\
&= \cos B \cos A \cos(180 - (A+B)) \\
&= \cos B \cos A \cos C.
\end{aligned}$$

Thus, one of the angles must be $90°$. Since $a > b > c$, we know that $\measuredangle A = 90°$.

Since $\measuredangle A = 90°$, we have that

$$\sin A + \sin B + \sin C = 1 + \sin B + \sin(90 - B) = 1 + \sin B + \cos B.$$

From Cauchy-Schwarz, we have that

$$(\sin B + \cos B)^2 < (1+1)(\sin^2 B + \cos^2 B) \Rightarrow (\sin B + \cos B)^2 < 2.$$

Therefore, $\sin B + \cos B < \sqrt{2}$.

Through this result, we get that

$$\frac{4\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < \frac{4\sqrt{3}}{\sqrt{2}+1}(\sqrt{2}+1) = 4\sqrt{3}.$$

Weitzenbock's Inequality states that

$$4\sqrt{3} \leq \frac{a^2+b^2+c^2}{S}.$$

Since $a > b > c$, we do not have the equality case.

From (1) and the above, we have that

$$\frac{3\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < 4\sqrt{3} < \frac{a^2+b^2+c^2}{S} = \frac{8R^2}{S}.$$

To conclude, $\frac{3\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < \frac{8R^2}{S}$.

Footnote by AEs: One Proof of Weitzenbock's Inequality

By Heron's Formula, we have that $S = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$. Thus,

$$\begin{aligned} S &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \\ &= \frac{1}{4}\sqrt{(a^2+b^2-c^2+2ab)\cdot[-(a^2+b^2-c^2-2ab)]} \\ &= \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+a^2c^2)-(a^4+b^4+c^4)}. \end{aligned}$$

Via AM-GM, we have that

$$\frac{a^4+b^4}{2}+\frac{b^4+c^4}{2}+\frac{a^4+c^4}{2} \geq a^2b^2+b^2c^2+a^2c^2.$$

Multiplying both sides by 4, we get that

$$4(a^4+b^4+c^4) \geq 4(a^2b^2+b^2c^2+a^2c^2).$$

This becomes

$$(a^2+b^2+c^2)^2 \geq 6(a^2b^2+b^2c^2+a^2c^2)-3(a^4+b^4+c^4).$$

Taking the square root of both sides, we have

$$a^2+b^2+c^2 \geq \sqrt{3}\cdot\sqrt{2(a^2b^2+b^2c^2+a^2c^2)-(a^4+b^4+c^4)} = 4\sqrt{3}\cdot S.$$

Equality occurs when $a=b=c$.

Problem 61B. Proposed by Alexandru Benescu

Let a, b and c be the sides of a triangle and R be the radius of its circumscribed circle. Prove that

$$(a+b+c)\cdot\sqrt[3]{(abc)^2} < 2R\cdot(ab+bc+ca).$$

Solution Problem 61B

Since $ab+bc+ca \geq 3\cdot\sqrt[3]{(abc)^2}$ through AM-GM, we have that

$$2R\cdot(ab+bc+ca) \geq 2R\cdot 3\sqrt[3]{(abc)^2}.$$

So, it is enough to prove that

$$2R\cdot 3\cdot\sqrt[3]{(abc)^2} > (a+b+c)\cdot\sqrt[3]{(abc)^2}.$$

We see that $a \leq 2R, b \leq 2R, c \leq 2R$. Furthermore, it is not possible for $a = 2R$, $b = 2R$ and $c = 2R$ at the same time. If it were the case, the center of the circumscribed circle would be on the three sides simultaneously, which is impossible.

Thus, $a+b+c<6R=2R\cdot 3$. Multiplying both sides of the inequality by $\sqrt[3]{(abc)^2}$, which is positive, we get

$$(a+b+c)\cdot\sqrt[3]{(abc)^2}<2R\cdot 3\cdot\sqrt[3]{(abc)^2}.$$

Therefore,

$$(a+b+c)\cdot\sqrt[3]{(abc)^2}<2R\cdot(ab+bc+ca).$$

Problem 62B. Proposed by Eliza Andreea Radu

If $a_1,a_2,\ldots,a_{2021}\in\mathbb{R}_+$ such that $\sum_{i=1}^{2021}a_i>2021$, prove that

$$a_1^{2^{2021}}\cdot 1\cdot 2+a_2^{2^{2021}}\cdot 2\cdot 3+\ldots+a_{2021}^{2^{2021}}\cdot 2021\cdot 2022>4086462.$$

Problem 63B. Proposed by Alexandru Benescu

Prove that

$$\frac{\sqrt{1^6+1}}{1^2}+\frac{\sqrt{2^6+1}}{2^2}+\frac{\sqrt{3^6+1}}{3^2}+\ldots+\frac{\sqrt{2020^6+1}}{2020^2}>\frac{\sqrt{(2021^2\cdot 1010)^2+2020^2}}{2021}.$$

Problem 64B. Proposed by Pavel Ciurea

In each 1×1 square of an $n\times n$ board, the number 1 or -1 is written. We define a move as the change of the signs of all the numbers on a diagonal. Determine for which configurations there is a finite series of moves that lead to the product of the numbers in each row and each column being 1.

Solution Problem 64B

We number the rows, top to bottom, from 1 to n and the columns, left to right, from 1 to n. We denote l_i as the product of the numbers in row i and c_i as the product of the numbers in column i, for all i from 1 to n.

We want to show that $l_i\cdot l_{n-i+1}\cdot c_i\cdot c_{n-i+1}$ remains constant after each move, for each $i\le\frac{n+1}{2}$.

To prove that $l_i\cdot l_{n-i+1}\cdot c_i\cdot c_{n-i+1}$ is invariant, it is enough to show that any diagonal intersects row i, row $n-i+1$, column i, and column $n-i+1$ an even number of times in total.

We let (a,b) represent the square in row a and column b. If we prove that each diagonal with one endpoint at square $(1,x)$ and the other at square $(x,1)$ intersects row i, row $n-i+1$, column i, and column $n-i+1$ an even number of times in total, then because of the symmetry, each diagonal will intersect row i, row $n-i+1$, column i, and column $n-i+1$ an even number of times overall.

If $1 \leq x < i$, then the considered diagonal row i, row $n-i+1$, column i, and column $n-i+1$ a total of 0 times.

If $i \leq x < n-i+1$, then the diagonal only intersects column i and row i, so it intersects row i, row $n-i+1$, column i, and column $n-i+1$ twice overall.

If $n-i+1 \leq x \leq n$, then the diagonal intersects row i, row $n-i+1$, column i, and column $n-i+1$, meaning that the diagonal intersects the relevant rows and columns 4 times overall.

Thus any diagonal intersects row i, row $n-i+1$, column i, and column $n-i+1$ an even number of times in total. Thus, $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1}$ remains constant after each move.

If n is odd then we have the case where $i = \frac{n+1}{2}$ results in row i being the same as row $n-i+1$ and column i being the same as column $n-i+1$. In this case using a similar argument to above, we obtain that $l_{\frac{n+1}{2}} \cdot c_{\frac{n+1}{2}}$ is invariant.

We want to show that if $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1} = 1$, for $\forall i$ where $1 \leq i \leq \frac{n+1}{2}$, then there is a series of moves after which the product of the numbers in each row and each column is 1 and if not, then such series does not exist.

If there is an i, where $1 \leq i \leq \frac{n+1}{2}$, such that $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1} = -1$, because $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1}$ remains constant after each move, at least one of l_i, l_{n-i+1},c_{n-i+1} and c_i equals -1 after any series of moves. So, there is no series of moves that will satisfy the condition.

We now want to show that if $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1} = 1$, for any i where $1 \leq i \leq \frac{n+1}{2}$, then there is a series of moves after which the product of the numbers in each row and each column is 1.

First, we want to show that there is a series of moves after which $c_i = 1$ for $\forall i$ where $1 \leq i \leq n$.

Let k be the smallest number such that $c_k = -1$. If such a k does not exist, then the above statement is obvious. We then consider the diagonal from square $(1, k)$ to $(1 + n - k, n)$ and apply a move on it. Thus, $c_i = 1$, for $\forall i$ where $1 \leq i \leq k$. After applying this move, we consider p, the new smallest number for which $c_p = -1$. We apply a move on the diagonal from square $(1, p)$ to square $(1 + n - p, n)$. By proceeding inductively, we find that $c_i = 1$ for $\forall i, 1 \leq i \leq n$.

From the fact that $l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1} = 1$ for $\forall i, 1 \leq i \leq \frac{n+1}{2}$ and that $c_i = 1$ for $\forall i, 1 \leq i \leq n \Rightarrow l_i = l_{n-i+1}$ and, if n is odd, we obtain that $l_{\frac{n+1}{2}} = c_{\frac{n+1}{2}} = 1$.

Let k be the largest number for which $l_k = -1$ and $1 \leq k \leq \frac{n}{2}$. If such k does not exist, then $l_i = 1$ for $\forall i, 1 \leq i \leq n$ and our proof would be finished. We apply a move on the diagonal from square $(k, 1)$ to square $(1, k)$ and apply a move on the diagonal from square $(n-k+1, 1)$ to square (n, k). Since both moves intersect columns 1 to k, we know that $c_i = 1$ for $\forall i, 1 \leq i \leq n$. Furthermore, $l_i = 1$ for $\forall i$ where $k \leq i \leq n-k+1$.

By proceeding inductively we obtain that $l_i = c_i = 1$ for $\forall i$ where $1 \leq i \leq n$.

Therefore, if

$$l_i \cdot l_{n-i+1} \cdot c_i \cdot c_{n-i+1} = 1, \text{ for } \forall i \text{ where } 1 \leq i \leq \frac{n+1}{2},$$

then there is a finite series of moves after which the product of the numbers in each row and each column is 1. If not, then there does not exist a series of moves that satisfies the problem.

Problem 65B. Proposed by Stefan-Ionel Dumitrescu

If $a, b, c \in \mathbb{R}_+$ such that $abc = a + b + c$ prove that

$$\sqrt{ab + bc + ca}\left(\frac{1}{3} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 4.$$

Solution Problem 65B

Lemma (Newton's Inequality)

If $a, b, c \in \mathbb{R}_+$, then

$$ab + ac + bc \geq \sqrt{3abc(a + b + c)}.$$

Proof of Newton's Inequality

By squaring, knowing that $a, b, c \in \mathbb{R}_+$, we have to prove that

$$\sum_{cyc} a^2b^2 + 2\sum_{cyc} a^2bc \geq 3\sum_{cyc} a^2bc,$$

or:

$$\sum_{cyc} a^2b^2 \geq \sum_{cyc} a^2bc.$$

The relationship is known and $a^2 + b^2 + c^2 \geq ab + ac + bc$ can be proved in many ways, such as Mean Inequality, Muirhead Theorem, or building squares. Equality occurs when $ab = ac = bc$, which means that $a = b = c$.

Addition from AEs:

One way to prove the inequality is to use Trivial Inequality. Since

$$\frac{1}{2}a^2(b-c)^2 + \frac{1}{2}b^2(c-a)^2 + \frac{1}{2}c^2(a-b)^2 \geq 0,$$

we get that $a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + ab^2c + abc^2$ via expansion.

Proof of Initial Inequality

Through the Lemma, we obtain

$$ab + ac + bc \geq \sqrt{3abc(a+b+c)}.$$

Since we are given that $abc = a + b + c$, we have that

$$\sqrt{3abc(a+b+c)} = \sqrt{3a^2b^2c^2}.$$

Thus, we get

$$\sqrt{ab+ac+bc} \geq \sqrt[4]{3a^2b^2c^2}. \ (1)$$

Using $GM - HM$, we obtain:

$$\sqrt[4]{3 \cdot a^2 \cdot b^2 \cdot c^2} \geq \frac{4}{\frac{1}{3} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}.$$

Rearranging the inequality, we get

$$\frac{1}{3} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{4}{\sqrt[4]{3 \cdot a^2 \cdot b^2 \cdot c^2}}. \ (2)$$

Multiplying (1) and (2), we get:

$$\sqrt{ab+ac+bc}\left(\frac{1}{3} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 4.$$

From the Lemma, the condition for equality is $a = b = c$. From $GM - HM$, the condition for equality is $\sqrt{3} = a = b = c$. From $abc = a + b + c$, the condition presented in the problem, we have that $a = b = c = \frac{abc}{3}$. Consequently, $a = b = c = \sqrt{3}$.

Problem 66B. Proposed by Stefan-Ionel Dumitrescu

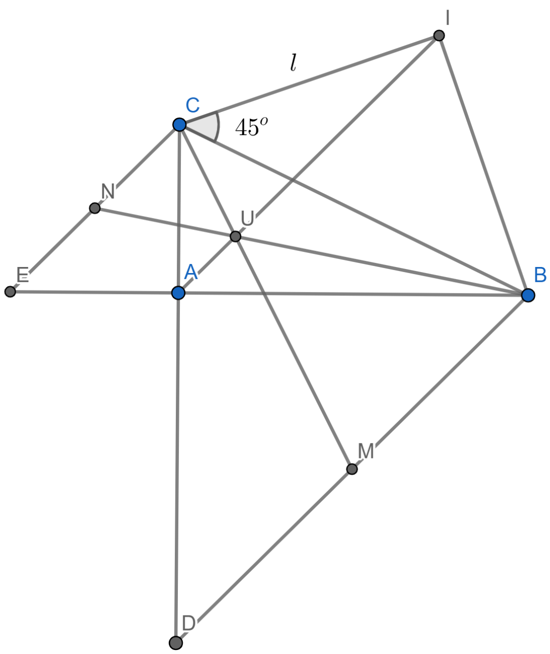
Consider right-angled triangle ABC with $m(\measuredangle BAC) = 90°$. Extend AC to D, where D and C are on different sides of AB, such that $AB = AD$. Similarly, extend AB to E, where E and B are on different sides of AC, such that $AC = AE$. Let M be the midpoint of BD, N be the midpoint of CE, and $BN \cap CM = \{U\}$. Take $I \in (AU)$, where A and I are on different sides of BC, such that $m(\measuredangle ICB) = 45°$. Calculate $m(\measuredangle CAI)$.

Solution Problem 66B

Vecten Point Theorem

For any $\triangle ABC$, three squares are constructed externally on the sides: π_1 on BC, π_2 on CA, and π_3 on AB. If O_1, O_2 and O_3 are the centers of squares π_1, π_2, π_3, respectively, then AO_1, BO_2 and CO_3 are concurrent at a point V called the outer Vecten point. Concurrency can be proven using Ceva's Theorem and Area Principle.

Solution Using Vecten Point Theorem



Since $AC = AE$, $AB = AD$, and $\measuredangle CAE = \measuredangle BAD = 90°$, we have that

$$\measuredangle AEC = \measuredangle ACE = \measuredangle ABD = \measuredangle ADB = 45°.$$

Thus, BD is a diagonal of the square constructed externally on AB and CE is a diagonal of the square constructed externally on AC.

Because M is midpoint of BD and N is the midpoint of CE, we observe that M is the center of the square build externally on side AB and N is the center of the square build externally on the side AC.

Consequently, U is the outer Vecten point for $\triangle ABC$.

Since $I \in (AU)$ and $m(\measuredangle ICB) = 45°$, the angle between one diagonal of a square and its side, we conclude that I must be the center of the square build externally on side BC. Thus, $m(\measuredangle IBC) = 45°$ and $m(\measuredangle BIC) = 90°$.

We have that

$$m(\measuredangle BAC) + m(\measuredangle BIC) = 90° + 90° = 180°,$$

meaning that $ABIC$ is a cyclic quadrilateral.

From inscribed angles, we have that $m(\measuredangle CAI) = m(\measuredangle IBC) = 45°$.

Problem 67B. Proposed by Stefan-Ionel Dumitrescu

Consider a cube $ABCDA'B'C'D'$. Point X lies on face $ADD'A'$. Point Y lies on face $ABB'A'$. Point W is a randomly chosen point on the edges, faces, or interior of the cube. If Z is the midpoint of XY, find the probability that W is the midpoint of an AZ.

Problem 68B. Proposed by Vedaant Srivastava

Determine the smallest positive integer M that satisfies the following two conditions:

1. M is a multiple of 2021.
2. For any positive integer n, if $n^n - 1$ is divisible by M, then $n - 1$ is divisible by M.